On Quasi -r-Normal Spaces

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Abstract: In this paper, we introduce the concept of quasi-r-normal spaces in topological spaces by using regular open sets in topological spaces and obtain some characterizations and preservation theorems for πgr-closed sets.

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INTRODUCTION

In 1968, Zaitsev [8] introduced the concept of quasi normal space in topological spaces and obtained several properties of such a space. Sadeq Ali Saad et al.,[7] introduced the concept of quasi p-normal spaces by using p-open sets and obtained its characterization.

In this paper, we use πgr-open sets to obtain the characterization of quasi-r-normal spaces.

PRELIMINARIES

Throughout this paper, spaces (X,τ) and (Y,σ)(or simply X or Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by Cl(A) and Int(A) respectively.

Definition 2.1: A subset A of a topological space X is said to be
1. regular open[5] if A = int (cl(A)).
2. π-open [8] if A is the finite union of regular open sets.

The complement of the above defined open sets are their respective closed sets in X.

Closure(r-closure) of A is the intersection of all closed (regular closed) sets containing A and Interior of A is the union of all open sets contained in A.

Definition 2.2: A subset A of X is called
1. g-closed[2] if cl (A)⊂U whenever A⊂U and U is open in X.
2. g*r-closed[5] if rcl(A)⊂U whenever A⊂U and U is open.
3. πg-closed[1] if cl(A)⊂ whenever A⊂U and U is π-open.
4. πgr-closed[3] if rcl(A)⊂ whenever A⊂U and U is π-open.

Definition 2.4: A topological space X is said to be
1. a normal space [7] if for every pair of disjoint closed subsets H and K, there exists disjoint open sets U and V of X such that H⊂U and K⊂V.
2. a quasi normal[7] if for every pair of disjoint π-closed subsets H and K, there exists disjoint open sets U and V of X such that H⊂U and K⊂V.
3. mildly normal[1] if for every pair of disjoint regular closed subsets H and K, there exists disjoint open sets U and V of X such that H⊂U and K⊂V.

Theorem 2.5: A subset A of a topological space X is πgr-open iff F⊂rint(A) whenever F is π-closed and F⊂A.
Definition 2.6: A function \( f: X \rightarrow Y \) is said to be
1. Almost closed [4][g-r-closed, πgr-closed] if \( f(F) \) is regular-closed (g*r-closed, πgr-closed) in \( Y \) for every closed set \( F \) of \( X \).
2. rc-preserving [4][almost g*r-closed, almost πgr-closed] if \( f(F) \) is regular closed(g*r- closed, πgr-closed) in \( Y \) for every regular closed set \( F \) of \( X \).
3. continuous[2][resp. almost continuous[6], π-continuous[1]] if \( f^1(F) \) is closed (resp. regular closed, \( π \)-closed) in \( X \) for every closed set \( F \) of \( Y \).
4. πgr-continuous [3] if \( f^1(F) \) is πgr-closed in \( X \) for every closed set \( F \) of \( Y \).

QUASI R-NORMAL SPACES.

Definition 3.1: A topological space \( X \) is said to be \( r \)-normal if for every pair of disjoint closed subsets \( H \) and \( K \), there exists disjoint regular open sets \( U \) and \( V \) of \( X \) such that \( H \subseteq U \) and \( K \subseteq V \).

Definition 3.2: A topological space \( X \) is said to be quasi \( r \)-normal (quasi regular normal) if for every pair of disjoint \( π \)-closed subsets \( H \) and \( K \), there exists disjoint regular open sets \( U \) and \( V \) of \( X \) such that \( H \subseteq U \) and \( K \subseteq V \).

Theorem 3.3: The following are equivalent for a space \( X \).

a) \( X \) is quasi \( r \)-normal.
b) For any disjoint \( π \)-closed sets \( H \) and \( K \), there exists disjoint g*r-open sets \( U \) and \( V \) such that \( H \subseteq U \) and \( K \subseteq V \).
c) For any disjoint \( π \)-closed sets \( H \) and \( K \), there exists disjoint g*r-open sets \( U \) and \( V \) such that \( H \subseteq U \) and \( K \subseteq V \).
d) For any \( π \)-closed set \( H \) any \( π \)-open set \( V \) containing \( H \), there exists an g*r-open set \( U \) of \( X \) such that \( H \subseteq U \subseteq rcl(U) \subseteq V \).
e) For any \( π \)-closed set \( H \) and \( π \)-open set \( V \) containing \( H \), there exists an \( π \)-gr-open set \( U \) of \( X \) such that \( H \subseteq U \subseteq rcl(U) \subseteq V \).

Proof: (a) \( \Rightarrow \) (b): Let \( X \) be quasi \( r \)-normal. Let \( H \) and \( K \) be disjoint \( π \)-closed sets in \( X \). By assumption, there exists disjoint regular open sets \( U, V \) such that \( H \subseteq U \) and \( K \subseteq V \). Since every regular open set is g*r-open, \( U \), \( V \) are g*r-open sets such that \( H \subseteq U \) and \( K \subseteq V \).

(b) \( \Rightarrow \) (c): Obvious.

e) \( \Rightarrow \) (d): Let \( H \) be any \( π \)-closed set and \( V \) be any \( π \)-open set containing \( H \). By assumption, there exists \( π \)-gr-open sets \( U \) and \( W \) such that \( H \subseteq U \) and \( X \setminus V \subseteq W \). By theorem 2.5, we get \( A \subseteq rcl(U), X \setminus rcl(W) \subseteq U \) and \( rcl(U) \cap rcl(U) = \phi \). Hence \( H \subseteq U \subseteq rcl(U) \subseteq X \setminus rcl(W) \subseteq V \).

d) \( \Rightarrow \) (e): Obvious.

(e) \( \Rightarrow \) (a): Let \( H, K \) be two disjoint \( π \)-closed sets of \( X \). Then \( H \subseteq X \setminus K \) and \( X \setminus K \) are \( π \)-open. By assumption there exists \( π \)-gr-open set \( G \) of \( X \) such that \( H \subseteq G \subseteq rcl(G) \subseteq X \setminus K \). Put \( U = r \text{ int } G \), \( V = X \setminus rcl(G) \). Then \( U \) and \( V \) are disjoint regular open sets of \( X \) such that \( H \subseteq U \) and \( K \subseteq V \).

Definition 3.4: A topological space \( X \) is said to be mildly \( r \)-normal if for every pair of disjoint regular closed sets \( H \) and \( K \) of \( X \), there exists disjoint regular open sets \( U \) and \( V \) of \( X \) such that \( H \subseteq U \) and \( K \subseteq V \).

Theorem 3.5: The following are equivalent for a space \( X \).

a) \( X \) is mildly \( r \)-normal.
b) For any disjoint regular closed sets \( H \) and \( K \), there exists disjoint g*r-open sets \( U \) and \( V \) such that \( H \subseteq U \) and \( K \subseteq V \).
c) For any disjoint regular closed sets \( H \) and \( K \), there exists disjoint g*r-open sets \( U \) and \( V \) such that \( H \subseteq U \) and \( K \subseteq V \).
d) For any regular closed set \( H \) any regular open set \( V \) containing \( H \), there exists an g*r-open set \( U \) of \( X \) such that \( H \subseteq U \subseteq rcl(U) \subseteq V \).
e) For any regular closed set \( H \) and each regular open set \( V \) containing \( H \), there exists a \( π \)-gr-open set \( U \) of \( X \) such that \( H \subseteq U \subseteq rcl(U) \subseteq V \).

Proof: Similar to that of above theorem 3.3.

Theorem 3.6: A surjection \( f: X \rightarrow Y \) is almost \( π \)-gr-closed iff for each subset \( S \) of \( Y \) and each regular open set \( U \) of \( X \) containing \( f^1(S) \), there exists a \( π \)-gr-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^1(V) \subseteq U \).

Proof: Necessity: Suppose that \( f \) is almost \( π \)-gr-closed. Let \( S \) be a subset of \( Y \) and \( U \) a regular open set containing \( f^1(S) \). If \( V = Y \setminus f(X \setminus U) \), then \( V \) is a \( π \)-gr-open set of \( Y \) such that \( S \subseteq V \) and \( f^1(V) \subseteq U \).
Sufficiency: Let F be any regular closed set of X. Then $\phi^1(Y - f(F)) \subset X - F$ and $X - F$ is regular open in X. There exists a $\pi gr$-open set $V$ of Y such that $Y - f(F) \subset V$ and $\phi^1(V) \subset X - F$. Therefore, we have $Y - V \subset f(F)$ and $f(F) \subset \phi^1(Y - V)$. Hence we obtain $f(F) = Y - V$ and $f(F)$ is $\pi gr$-closed in Y which shows that $f$ is almost $\pi gr$-closed.

Preservation theorems:

**Theorem 3.7:** If $f: (X, \tau) \rightarrow (Y, \sigma)$ is an almost $\pi gr$-continuous $\pi$-closed injection and Y is quasi -r-normal, then X is quasi-r-normal.

**Proof:** Let A and B be any two disjoint $\pi$-closed sets of X. Since $f$ is a $\pi$-closed injection, $f(A)$ and $f(B)$ are disjoint $\pi$-closed sets of Y. Since Y is quasi-r-normal, there exists disjoint regular open sets G and H such that $f(A) \subset G$ and $f(B) \subset H$. Since $f$ is almost $\pi gr$-continuous, $\phi^1(G)$ and $\phi^1(H)$ are disjoint $\pi gr$-open sets containing A and B which shows that X is quasi - r-normal.

**Lemma 3.8:** A surjection $f: (X, \tau) \rightarrow (Y, \sigma)$ is rc-preserving iff for each subset S of Y and each regular open set U of X containing $\phi^1(S)$ there exists a regular open set V of Y such that $S \subset V$ and $\phi^1(V) \subset U$.

**Theorem 3.9:** If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a $\pi$-continuous, rc-preserving surjection and X is quasi-r-normal, then Y is r-normal.

**Proof:** Let A and B be any two disjoint closed sets of Y. Then $\phi^1(A)$ and $\phi^1(B)$ are disjoint $\pi$-closed sets of X. Since X is quasi-r-normal, there exists disjoint regular open sets G and H such that $\phi^1(A) \subset G$ and $\phi^1(B) \subset H$. Set $K = Y - f(X - G)$ and $L = Y - f(X - H)$. By lemma 3.8, K and L are regular open sets of Y such that $A \subset K$, $B \subset L$, $\phi^1(K) \subset G$, $\phi^1(L) \subset H$. Since G and H are disjoint and so K and L. Since K and L are regular open, we obtain $A \subset \text{rint}(K)$, $B \subset \text{rint}(L)$ and $\text{rint}(K) \cap \text{rint}(L) = \emptyset$. Therefore Y is r-normal.

**Theorem 3.10:** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a $\pi$-irresolute, almost $\pi gr$-closed surjection .If X is a quasi-r-normal space, then Y is quasi-r-normal.

**Proof:** Let A and B be any two disjoint $\pi$-closed sets of Y. Since $f$ is $\pi$-irresolute, $\phi^1(A)$ and $\phi^1(B)$ are disjoint $\pi$-closed subsets of X. Since X is quasi-r-normal, there exists regular open sets G and H of X such that $\phi^1(A) \subset G$ and $\phi^1(B) \subset H$. By theorem 3.7, there exists $\pi gr$-open sets K and L of such that $A \subset K$ and $B \subset L$, $\phi^1(K) \subset G$, $\phi^1(L) \subset H$. Since G and H are disjoint, so $\text{rint}(K) \cap \text{rint}(L) = \emptyset$. Therefore, U is quasi -r-normal.

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