Observation on the Binary Quadratic Equation $3x^2-8xy+3y^2+2x+2y+6=0$

S. Vidhyalakshmi, M. A. Gopalan*, K. Lakshmi
Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, Tamilnadu, India

*Corresponding Author:
M. A. Gopalan
Email: ravilgopalan@gmail.com

Abstract: The binary quadratic equation $3x^2-8xy+3y^2+2x+2y+6=0$ is studied for its non-trivial integral solutions. The recurrence relations satisfied by the solutions $x$ and $y$ are given. A few interesting properties among the solutions are presented.

Keywords: Binary quadratic equation, Integral solutions.

MSC subject classification: 11D09.

INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-6]. In [7-14] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions.

However, in [13] it is shown that the hyperbola represented by $3x^2 + xy = 14$ has only finite number of integral points. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting binary quadratic equation given by $3x^2-8xy+3y^2+2x+2y+6=0$. The recurrence relations satisfied by the solutions $x$ and $y$ are given. Also a few interesting properties among the solutions are exhibited.

METHOD OF ANALYSIS

The Diophantine equation under consideration is

$$3x^2-8xy+3y^2+2x+2y+6=0 \quad (1)$$

It is to be noted that (1) represents a hyperbola. By shifting the origin to the centre $(1,1)$, (1) reduces to

$$3X^2-8XY+3Y^2=-8 \quad (2)$$

where $x = X + 1, \ y = Y + 1 \quad (3)$

Again setting

$$X = M + N, Y = M - N \quad (4)$$

in (2) it simplifies to the equation

$$M^2 = 7N^2 + 4 \quad (5)$$

Now, consider the Pellian equation

$$M^2 = 7N^2 + 1 \quad (6)$$

whose general solution $\left( \overset{\sim}{N}_n, \overset{\sim}{M}_n \right)$ is given by
$$N_n = \frac{1}{2\sqrt{7}} \left[ (8 + 3\sqrt{7})^{n+1} - (8 - 3\sqrt{7})^{n+1} \right],$$

$$M_n = \frac{1}{2} \left[ (8 + 3\sqrt{7})^{n+1} + (8 - 3\sqrt{7})^{n+1} \right].$$

Thus, the general solution \( \left( N_n, M_n \right) \) of (5) is given by

$$N_n = 2N_n = \frac{1}{\sqrt{7}} \left[ (8 + 3\sqrt{7})^{n+1} - (8 - 3\sqrt{7})^{n+1} \right]$$

$$M_n = 2M_n = \left[ (8 + 3\sqrt{7})^{n+1} + (8 - 3\sqrt{7})^{n+1} \right].$$

Taking advantage of (3) and (4), the sequence of integral solutions of (1) can be written as

$$x_n = M_n + N_n + 1 = 2M_n + 2N_n + 1$$

$$y_n = M_n - N_n + 1 = 2M_n - 2N_n + 1, \quad n = 0, 1, 2 \ldots \ldots .$$

Thus (7) and (8) represent the non-zero distinct integral solutions of (1).

The above values of \( x_n \) and \( y_n \) satisfy respectively the following recurrence relations.

$$x_{n+2} - 16x_{n+1} + x_n = -14,$$

$$y_{n+2} - 16y_{n+1} + y_n = -14, \quad n = 0, 1, 2 \ldots \ldots .$$

A few numerical examples are given below

<table>
<thead>
<tr>
<th>n</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>23</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>351</td>
<td>159</td>
</tr>
<tr>
<td>2</td>
<td>5579</td>
<td>2519</td>
</tr>
<tr>
<td>3</td>
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<td>40131</td>
</tr>
<tr>
<td>4</td>
<td>1416791</td>
<td>639563</td>
</tr>
</tbody>
</table>

Some relations satisfied by the solutions (7) and (8) are as follows:

1. Both the values of \( x, y \) are positive and odd.
2. \( 18x_n - 8y_n - 2y_{n+1} \equiv 0 \pmod{8} \)
3. \( 20x_n - 9y_n - x_{n+1} \equiv T_{10,2} \)
4. \( 2 (x_{3n+2} + 3x_{n+3} + 3y_{n+3}) = (x_n + y_n - 2) (29y_{2n+2} - 13x_{2n+2} - 12) \)
5. \( 2 (29y_{3n+3} - 13x_{3n+3} + 87y_{n+1} - 39x_{n+1} - 64) - (x_n + y_n - 2) (29y_{2n+2} - 13x_{2n+2} - 12) = 0 \)
6. \( (x_n + y_n - 2y_{n+1} - 13x_{3n+3} + 87y_{n+1} - 39x_{n+1} - 64) - 2(x_{3n+3} + 3y_{n+3} + 4x_{n+1} + 4y_{n+1} + 2) = 0 \)
7. \( (x_{2n+1} + 2y_{n+1} + 2) (29y_{2n+2} - 13x_{2n+2} - 12) = 2(29y_{4n+4} - 13x_{4n+4} + 116y_{2n+2} - 52x_{2n+2} - 68) \)
8. \( 8.28(x_{2n+1} - y_{2n+2})^2 - (x_{2n+1} + y_{2n+2} - 6)(x_n + y_n - 2) = 0 \)
9. Each of the following is a nasty number:

\( 3(x_{2n+1} + y_{2n+1} + 2) \)
Remarkable observations

I. By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the Parabola.

(a) Illustration 1: It is to be noted that the Parabola

\[ Y^2 = 2X \]

is satisfied for the following three sets of values of \( X \) and \( Y \)

Set1:

\[
Y = 29y_{n+1} - 13x_{n+1} - 16 \\
X = x_{2n+1} + y_{2n+1} + 2
\]

Set2:

\[
Y = 29y_{2n+2} - 13x_{2n+2} - 12 \\
X = 29y_{4n+4} - 13x_{4n+4} + 116y_{2n+2} - 52x_{2n+2} - 68
\]

Set3:

\[
Y = x_{2n+1} + y_{2n+1} + 2 \\
X = x_{4n+3} + y_{4n+3} + 4x_{2n+1} + 4y_{2n+1} + 2
\]

(b) Illustration 2: The Parabola

\[ 7Y^2 = 2X \]

is satisfied for the following set of values of \( X \) and \( Y \)

\[
Y = 5x_{n+1} - 11y_{n+1} + 6 \\
X = x_{2n+1} + y_{2n+1} - 6
\]

II. If \((x_0, y_0)\) is any given solution of (1), then each of the following expressions satisfies (1):

\[
(-y_0 + 2, -x_0 + 2), (-9x_0 + 4y_0 + 6, -20x_0 + 9y_0 + 12).
\]

CONCLUSION

In conclusion one may search for other patterns of solutions and their corresponding properties.

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REFERENCES