The exact single traveling wave solutions to the Gardner equation

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Abstract: Using the complete discrimination system for polynomial, we give the classification of single traveling wave solutions to the Gardner equation.

Keywords: Gardner equation; single traveling wave solution; complete discrimination system for polynomial

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INTRODUCTION
In the paper, we consider the Gardner equation

\[ u_t = (\beta w - \frac{B^2}{2} u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx}, \] (1)

where \( w_x = v_y \). We will give the classification of single traveling wave solutions to the Gardner equation by complete discrimination system for polynomial method.

EXACT SOLUTIONS
In order to obtain the exact traveling wave solutions, we take a wave transformation \( u = u(\xi) \) and \( \xi = k_1 x + k_2 y + ct \). As \( w_x = v_y \), we have \( w = \frac{k_1}{k_2} u + G_1 \) \((G_1 \) is an integral constant). The Gardner equation is reduced to the following ODE,

\[ cu'' = \beta k_2 u u' + \beta G_1 k_2 u' - \frac{B^2}{2} k_1 u^2 u' + k_1 \delta u u' + \frac{k_1^2}{k_1} u' + \epsilon^2 k_1^3 u''' \] (2)

Multiplying the both sides of the Eq.(3) by \( u'^2 \) and integrating it once, we can have:

\[ (u')^2 = a_1 u^3 + a_2 u^3 + a_3 u^2 + a_0, \] (3)

where \( a_1 = \frac{\beta^2}{6 \epsilon^2 k_1^4}, a_2 = -\frac{2(\beta k_2 \epsilon k_1 - k_1^2 k_2)}{3 \epsilon^2 k_1^2}, a_3 = \frac{2 k_1 (\beta k_2 \epsilon k_1 - k_1^2 k_2)}{\epsilon^2 k_1^4}, a_0 = G_2 \). \( G_2 \) is an integral constant. The solutions of \( u \) can be given from

\[ \pm (\xi - \xi_0) = \int \frac{du}{\sqrt{a_1 u^3 + a_2 u^3 + a_3 u^2 + a_0}}. \] (4)

In order to solve the Eq.(4), we take the transformation as \( y = (a_4)^{\frac{3}{2}} (u + \frac{\alpha_4}{a_4}) \) when \( a_4 > 0 \), then the Eq.(4) yields

\[ \pm (a_4)^{\frac{3}{2}} (\xi - \xi_0) = \frac{dy}{\sqrt{y^3 + p y^2 + q y + r}}, \] (5)

where \( p = \frac{\alpha_4}{a_4}, q = (\frac{a_1}{8 a_4} - \frac{a_2}{2 a_4})(a_4)^{-\frac{3}{2}}, r = -\frac{3 a_1}{2 a_4} + \frac{a_2}{16 a_4} + a_0 \). If \( a_4 < 0 \), then we take the transformation as \( y = (-a_4)^{\frac{3}{2}} (u + \frac{\alpha_4}{a_4}) \), the Eq.(4) becomes
\begin{align}
\pm(a_4)^{\frac{1}{4}} (\xi - \xi_0) = \frac{dy}{\sqrt{-(y^4 + py^2 + qy + r)}},
\end{align}

where \( p = \frac{-a_3}{a_4} \), \( q = (-\frac{a_1}{8a_4} + \frac{a_2a_4}{2a_1})(-a_4)^{\frac{1}{4}} \), \( r = \frac{a_1^2}{256a_4^2} - \frac{a_2a_4}{16a_1} - a_0 \). To get the solutions to the Eq.(5) and (6), we denote \( F(y) = y^4 + py^2 + qy + r \). Its complete discrimination system \([5,6]\) is computed as follows:

\begin{align}
D_1 &= 4, D_2 = -p, D_3 = -2p^3 + 8pr - 9q^2, \\
D_4 &= -p^3q^2 + 4p^4r + 36pq^2r - 32p^2r^2 - \frac{27}{4}q^4 + 64r^3, \\
E_2 &= 9p^2 - 32pr.
\end{align}

According to the complete discrimination system for polynomial \( F(w) \), the classification of the traveling wave solutions of the Gardner equation can be discussed:

Case 1. \( D_2 = 0, D_3 = 0 \) and \( D_4 = 0 \). Then we have \( F(y) = y^4 \), here \( a_4 > 0 \). By Eq.(5), we can give the solutions

\begin{align}
y = -(a_4)^{\frac{1}{4}} (\xi - \xi_0)^{-1}.
\end{align}

Case 2. \( D_2 < 0, D_3 = 0 \), and \( D_4 = 0 \). \( F(y) = ((y-l)^2 + s^2)^2 \), where \( l, s \) are real numbers, and \( s > 0 \). For \( a_4 > 0 \), we have

\begin{align}
y = s \tan((a_4)^{\frac{1}{4}} (\xi - \xi_0)s) + l.
\end{align}

Case 3. \( D_2 > 0, D_3 = 0, D_4 = 0 \) and \( E_2 > 0 \). Then we have \( F(y) = (y - \alpha)^2(y - \beta)^2 \), where \( \alpha, \beta \) are real numbers, and \( \alpha \neq \beta \). For \( a_4 > 0 \), we have

\begin{align}
\pm(a_4)^{\frac{1}{4}} (\xi - \xi_0) = \frac{1}{\alpha - \beta} \ln \left| \frac{y - \alpha}{y - \beta} \right|.
\end{align}

For \( y > \alpha \) or \( y < \beta \), by the Eq.(10)we have

\begin{align}
y = \beta + \frac{\beta - \alpha}{\exp[(a_4)^{\frac{1}{4}}((\alpha - \beta)(\xi - \xi_0))] - 1},
\end{align}

when \( \beta < y < \alpha \), by the Eq.(10)we have

\begin{align}
y = \beta - \frac{\beta - \alpha}{\exp[(a_4)^{\frac{1}{4}}((\alpha - \beta)(\xi - \xi_0))] + 1}.
\end{align}

Case 4. \( D_2 > 0, D_3 = 0, D_4 = 0 \) and \( E_2 = 0 \). Then we have \( F(y) = (y - \alpha)^3(y - \beta) \), where \( \alpha, \beta \) are real numbers, and \( \alpha \neq \beta \). When \( a_4 > 0 \), we have

\begin{align}
\pm(a_4)^{\frac{1}{4}} (\xi - \xi_0) = \frac{2}{\beta - \alpha} \sqrt{\frac{y - \beta}{y - \alpha}}.
\end{align}

When \( y > \alpha \), \( y > \beta \) or \( y < \alpha \), \( y < \beta \), the solution is

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\[ y = \alpha + \frac{4(\alpha - \beta)}{(a_4)^{\frac{1}{4}}(\alpha - \beta)^2(\xi - \xi_0)^2 - 4}. \]  

(16)

For \( a_4 < 0 \), we have

\[ \pm(a_4)^{\frac{1}{4}}(\xi - \xi_0) = \frac{2}{\alpha - \beta} \sqrt{\frac{\beta - y}{y - \alpha}}. \]  

(17)

When \( y > \alpha \), \( y < \beta \) or \( y < \alpha \), \( y > \beta \), we can get a solitary solution as

\[ y = \alpha - \frac{4(\alpha - \beta)}{(a_4)^{\frac{1}{4}}(\alpha - \beta)^2(\xi - \xi_0)^2 + 4}. \]  

(18)

Case 5. \( D_2 D_3 < 0 \), and \( D_4 = 0 \). \( F(y) = (y - \alpha)((y - l)^2 + s^2) \). By Eq.(5), we have

\[ y = \frac{[e^{\pm(a_4)^{\frac{1}{4}}m(\xi - \xi_0)} - \frac{\alpha - 2l}{m}]}{[e^{\pm(a_4)^{\frac{1}{4}}m(\xi - \xi_0)} - \frac{\alpha - 2l}{m}]^2 - 1}, \]  

(19)

where \( m = \sqrt{\alpha - l)^2 + s^2} \).

Case 6. For \( D_3 > 0 \), \( D_1 > 0 \), \( D_4 > 0 \) and other cases, the corresponding solutions can be expressed by hyper-elliptic functions or hyper-elliptic integral. We omit them for simplicity.

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