Ascertain Subclasses of Meromorphically Multivalent Functions with Negative Coefficient Associated with Linear Operator

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Abstract: In this paper, we introduce the subclasses $A_{\lambda,p}^n(a,b,c;\alpha,A,B)$ and $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$ of meromorphic multivalent functions in the punctured unit disk $U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$ by using a differential operator $h_{\lambda,p}^n(a,b,c,z)f(z)$. We obtain coefficient estimates, distortion theorem, radius of convexity and closure Theorems for the class $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$.

Keywords: Meromorphic functions, $p$-valent starlikeness and convexity.

2010 Mathematics Subject Classifications: 30C45

INTRODUCTION

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1,2,3,...\}),$$

which are analytic in the punctured unit disk $U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = \mathbb{C} \setminus \{0\}$.

Also let $\Omega_p$ denote the subclass of $\sum_p$ of meromorphic multivalent functions in $U^*$, which have the power series representation

$$f(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} d_{k+p} z^{k+p} \quad (a_{k+p} \geq 0).$$

A function $f(z) \in \sum_p$ is said to be $p$-valent meromorphically starlike of order $\alpha$, if and only if

$$\Re\left\{ \frac{-zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U^*),$$

for some $\alpha (0 \leq \alpha < p)$. We denote the class of all meromorphic $p$-valent starlike functions of order $\alpha$ by $\sum_p(\alpha)$. Further a function $f(z)$ in $\sum_p$ is said to be meromorphic $p$-valent convex of order $\alpha$ if and only if

$$\Re\left\{ \frac{-zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U^*),$$

for some $\alpha (0 \leq \alpha < p)$. We denote the class of all meromorphic $p$-valent convex functions of order $\alpha$ by $K_p(\alpha)$.

The classes $\sum_p(\alpha)$ and $K_p(\alpha)$ and various other subclasses of $\sum_p$ have been studied rather extensively by Aouf et al. [1], [3] and [5], Joshi and Srivastava [6], Kulkarni et al. [7], Owa et al. [10], and others.

For $\alpha = 0$, we obtain the class $\sum(p)$ and $K(p)$ of meromorphic $p$-valent starlike and convex functions with respect to the origin.
Denote by $\Sigma_p^*(\alpha)$ and $K_p^*(\alpha)$ the classes obtained by considering intersection, respectively, of the classes $\Sigma_p(\alpha)$ and $K_p(\alpha)$ with $\Omega_p$, i.e.

$$\Sigma_p^*(\alpha) = \Sigma_p(\alpha) \cap \Omega_p; \quad (0 \leq \alpha < p)$$

$$K_p^*(\alpha) = K_p(\alpha) \cap \Omega_p; \quad (0 \leq \alpha < p)$$

(1.5)

The function $f(z)$ is said to be subordinate to $F(z)$, if there exists a function $w(z)$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. In particular, if $F$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

For $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

(1.6)

the Hadamard product (or convolution) of $f$ and $g$ is denoted by $(f \ast g)(z)$ and defined by

$$(f \ast g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{k+p} b_{k+p} z^{k+p} \quad (1.7)$$

For the function $f(z) \in \Sigma_p$, Aouf [2] define the following differential operator

$$S^0_{\lambda,p}f(z) = f(z)$$

$$S^1_{\lambda,p}f(z) = (1-\lambda)f(z) \ast \frac{\lambda}{z} f'(z) + \frac{2\lambda}{z^p}$$

$$= \frac{1}{z^p} + \sum_{k=0}^{n} \left( 1 + \frac{\lambda k}{p} \right) a_{p+k} z^{p+k} = S_{\lambda,p}f(z). \quad (\lambda \geq 0, p \in \mathbb{N})$$

(1.8)

$$S^2_{\lambda,p}f(z) = S_{\lambda,p}(D^1_{\lambda,p}f(z)),$$

$$S^n_{\lambda,p}f(z) =\cdots = S_{\lambda,p}(S^{n-1}_{\lambda,p}f(z))$$

$$= (1-\lambda)S^{n-1}_{\lambda,p}f(z) \ast \frac{\lambda}{p} z \left( S^{n-1}_{\lambda,p}f(z) \right)' + \frac{2\lambda}{z^p} (\lambda \geq 0; n, p \in \mathbb{N}).$$

(1.9)

It can be easily seen that

$$S^n_{\lambda,p}f(z) = \frac{1}{z^p} + \sum_{k=0}^{n} \left( 1 + \frac{k \lambda}{p} \right)^n a_{p+k} z^{p+k} \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p \in \mathbb{N}).$$

(1.10)

For positive numbers $a$, $b$ and $c$, define the operator $I_p(a,b;c,z)$ by

$$I_p(a,b;c,z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left( \frac{a}{c} \right)_k \frac{(b)_k}{k!} z^{p+k}. \quad (1.10)$$

With the aid of the operator $S^n_{\lambda,p}f(z)$ defined by (1.9) and the operator $I_p(a,b;c,z)$, defined by (1.10) we define the operator $h^n_{\lambda,p}(a,b;c,z)f(z)$ in terms of the hadmered product or (convolution) by

$$h^n_{\lambda,p}(a,b;c,z)f(z) = S^n_{\lambda,p}f(z) \ast I_p(a,b;c,z), \quad (1.11)$$

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which can be written for \( f(z) \) defined by (1.1) as

\[
h_{n,p}(a,b;c,z) f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} Q^*_{k,\lambda}(a,b;c) a_{p+k} z^{p+k}.
\]  

(1.12)

for \( n \in \mathbb{N} , \lambda \geq 0 \),

where

\[
Q^*_{k,\lambda}(a,b;c) = (1 + \frac{k\lambda}{p}) a_k (b)_k (c)_k (1)_k.
\]  

(1.13)

With the aid of the differential operator \( h_{n,p}(a,b;c,z) f(z) \) we define the following subclasses of multivalent and meromorphic functions.

**Definition 1.** A function \( f(z) \in \sum_p \) defined by (1.1) is said to be in the class \( A^n_{\alpha,p} (a,b,c;\alpha,A,B) \) if it satisfies the following subordination condition:

\[
1 + \frac{z(h_{n,p}(a,b;c,z) f(z))^n}{(h_{n,p}(a,b;c,z) f(z))'} p + \frac{[pB + (A - B)(p - \alpha)]z}{1 + Bz} (z \in U^*)
\]

(1.14)

or, equivalently, if the following inequality holds true:

\[
1 + \frac{(z(h_{n,p}(a,b;c,z) f(z))^n)}{(h_{n,p}(a,b;c,z) f(z))'} p 1 + \frac{p}{[1 + Bz]} (z \in U^*)
\]

(1.15)

Also let \( A^n_{\alpha,p} (a,b,c;\alpha,A,B) = A^n_{\alpha,p} (a,b,c;\alpha,A,B) \) if and only if

\[
0 \leq \alpha < P; -1 \leq A < B \leq 1; 0 < B \leq 1; p \in \mathbb{N} ; n \in \mathbb{N}_0; \lambda \geq 0 ; \delta \geq 0
\]

It may be noted that for suitable choice of \( \delta, A, B, n, p, \lambda \) and \( \alpha \) the class \( A^n_{\alpha,p} (a,b,c;\alpha,A,B) \) extends several classes of analytic and \( p \)-valent meromorphic functions such that Aouf and Shammaky[4], Srivastava et al. [11], Liu. and Srivastava ([8], [9]) and Uralegaddi and Ganigi [12].

**Basic properties of the class** \( A^n_{\alpha,p} (a,b,c;\alpha,A,B) \).

We first determine a necessary and sufficient condition for a function \( f(z) \in \Omega_p \) of the form (1.2) to be in the class \( A^n_{\alpha,p} (a,b,c;\alpha,A,B) \)

**Theorem 1.** Let the function \( f(z) \in \Omega_p \) defined by (1.2), the function \( f(z) \in A^n_{\alpha,p} (a,b,c;\alpha,A,B) \) if and only if

\[
\sum_{k=0}^{\infty} (k + p) Q^n_{k,\lambda}(a,b;c) M_k(\alpha,A,B,P) a_{k+p} \leq p(B - A)(p - \alpha)
\]

(2.1)

where

\[
M_k(\alpha,A,B,P) = \left( [k + p](B + 1) + p(A + 1) + (B - A)\alpha \right],
\]

(2.2)

and \( Q^n_{k,\lambda}(a,b;c) \) is given by (1.13).

**Proof.** Suppose that the function \( f(z) \in \Omega_p \) defined by (1.2) be in the class \( A^n_{\alpha,p} (a,b,c;\alpha,A,B) \), then from (1.15) we have

\[
\begin{aligned}
&\frac{(z(h_{n,p}(a,b;c,z) f(z))^n)}{(h_{n,p}(a,b;c,z) f(z))'} (1 + p) (h_{n,p}(a,b;c,z) f(z))' \\
&\frac{B(h_{n,p}(a,b;c,z))'}{B(h_{n,p}(a,b;c,z))'} + z(h_{n,p}(a,b;c,z))^n + [pB + (A - B)(p - \alpha)] h_{n,p}(a,b;c,z) f(z)'
\end{aligned}
\]

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\[ p(B - A)(p - \alpha) - \sum_{k=0}^{n} (k + p)(k + 2p)Q_{k,\lambda}^n(a, b, c) a_k z^k \leq \sum_{k=0}^{n} (k + p)(k + 2p)Q_{k,\lambda}^n(a, b, c) \left[ (k + p)B + (B - A)\alpha + AP \right] A_{k+p}, \]
\[ < 1 \quad (z \in U). \quad (2.3) \]
Since \( \Re \{z\} \leq 1 \), for any \( z \), choosing \( z \) to be real and letting \( z \to 1 \) through real value, then (2.3) yield
\[ \sum_{k=0}^{\infty} (k + p)(k + 2p)Q_{k,\lambda}^n(a, b, c) a_{k+p} \leq p(B - A)(p - \alpha) - \sum_{k=0}^{n} (k + p)Q_{k,\lambda}^n(a, b, c) \left[ (k + p)B + (B - A)\alpha + AP \right] A_{k+p}, \]
which leads us immediate to the coefficient inequality (2.1).

Next in order to prove the converse we assume that the inequality (2.1) holds true, then we observe that
\[ \left| \frac{(z(h^n_{\lambda,\lambda}f(z))^n + (1 + p)(h^n_{\lambda,\lambda}f(z))^n)}{B(h^n_{\lambda,\lambda}f(z))^n + (1 + p)(h^n_{\lambda,\lambda}f(z))^n} + \frac{pB + (A - B)(P - \alpha)}{h^n_{\lambda,\lambda}f(z)} \right| \]
\[ \leq \sum_{k=0}^{\infty} (k + p)(k + 2p)Q_{k,\lambda}^n(a, b, c) a_{k+p} \leq p(B - A)(P - \alpha) - \sum_{k=0}^{n} (k + p)Q_{k,\lambda}^n(a, b, c) \left[ (k + p)B + (B - A)\alpha + AP \right] A_{k+p}, \]
\[ < 1 \quad (z \in U). \quad (2.5) \]
Hence by maximum modulus theorem, we have \( f(z) \in A_{\lambda,\lambda}^n(a, b, c; \alpha, A, B) \). This completes the proof of Theorem.

**Corollary 1.** Let the function \( f(z) \in \Omega_p \) defined by (1.2), if \( f(z) \in A_{\lambda,\lambda}^n(a, b, c; \alpha, A, B) \) then
\[ d_{k+p} \leq \frac{p(B - A)(P - \alpha)}{(K + P)Q_{k,\lambda}^n(a, b, c)M_k(\alpha, A, B)} z^{k+p} \quad (k \geq 0, p \in N). \quad (2.6) \]
The result is sharp for the function \( f(z) \) given by
\[ f(z) = z^{-p} - \frac{p(B - A)(P - \alpha)}{(k + p)Q_{k,\lambda}^n(a, b, c)M_k(\alpha, A, B)} z^{k+p} \quad (k \geq 0, p \in N). \quad (2.7) \]

Next we prove the following distortion and growth properties for the class \( A_{\lambda,\lambda}^n(a, b, c; \alpha, A, B) \).

**Theorem 2.** If a function \( f(z) \in \Omega_p \) defined by (1.2) is in the class \( A_{\lambda,\lambda}^n(a, b, c; \alpha, A, B) \), then
\[ \left| \frac{(p + m - 1)!}{(p - 1)!} \frac{p!(B - A)(P - \alpha)}{(p - m)!M_\lambda(\alpha, A, B, p)} \right| r^{-p-m} \]
\[ \leq |f^n(z)| \leq \left| \frac{(p + m - 1)!}{(p - 1)!} + \frac{p!(B - A)(P - \alpha)}{(p - m)!M_\lambda(\alpha, A, B, p)} \right| r^{-p-m} \quad (0 < |z| = r < 1, 0 \leq m < p), \quad (2.8) \]
where the result is sharp for the function \( f(z) \) given by
\[ f(z) = z^{-p} - \frac{(B - A)(P - \alpha)}{M_\lambda(\alpha, A, B, p)} z^p \quad (p \in N), \quad (2.9) \]
and

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\( M_{\gamma}(\alpha,A,B,P) = \left[ p(B+1)+p(A+1)+(B-A)\alpha \right] \).

\((0 \leq \alpha < P; A+B \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; p \in N; n \in N_0; \lambda \geq 0; \delta \geq 0)\)

**Proof.** For \( f(z) \in A_{\alpha,p}^n (a,b,c;\alpha,A,B) \) we find from Theorem 1, that

\[
pQ_{0,k}^n (a,b;c)M_\alpha (\alpha,A,B,P) \sum_{k=0}^{\infty} a_{k+p} \leq \sum_{k=0}^{\infty} (k+p)Q_{0,k}^n (a,b;c)M_k (\alpha,A,B,P) a_{k+p}
\]

\[
\leq p(B-A)(P-\alpha).
\]

or

\[
\sum_{k=0}^{\infty} a_{k+1} \leq \frac{(B-A)(P-\alpha)}{Q_{0,k}^n (a,b;c)M_\alpha (\alpha,A,B,P)} = \frac{(B-A)(P-\alpha)}{M_\alpha (\alpha,A,B,P)}.
\]

(2.10)

Now by differentiating \( f(z) \) in (1.2) \( m \) times, we have

\[
f^m(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{p-m} - \sum_{k=0}^{\infty} \frac{(k+p)!}{(k+p-m)!} a_{k+p} z^{k+p-m},
\]

\((m \in N_0, P \in N, m < P)\)

Thus, for \( 0 \leq |z| = r < 1 \),

\[
|f^m(z)| = \left| (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{p-m} - \sum_{k=0}^{\infty} \frac{(k+p)!}{(k+p-m)!} a_{k+p} z^{k+p-m} \right|
\]

\[
\leq \frac{(p+m-1)!}{(p-1)!} r^{p-m} + \sum_{k=0}^{\infty} \frac{(k+p)!}{(k+p-m)!} a_{k+p} r^{k+p-m}
\]

\[
\leq \frac{(p+m-1)!}{(p-1)!} r^{p-m} + \frac{p!}{(p-m)!} \sum_{k=0}^{\infty} a_{k+p}
\]

\[
\leq \frac{(p+m-1)!}{(p-1)!} r^{p-m} + \frac{p!}{(p-m)!} \frac{(B-A)(P-\alpha)}{M_\alpha (\alpha,A,B,P)} r^{p-m},
\]

similarly

\[
\geq \frac{(p+m-1)!}{(p-1)!} r^{p-m} + \frac{p!}{(p-m)!} \frac{(B-A)(P-\alpha)}{M_\alpha (\alpha,A,B,P)} r^{p-m},
\]

The sharpness of each the inequality in (2.8) satisfies by the function \( f(z) \) given by (2.9).

Next we determine the radii of meromorphically \( p \)-valent starlikeness and convexity of order \( \gamma \) \((0 \leq \gamma < p)\) for functions in the class \( A_{\alpha,p}^n (a,b,c;\alpha,A,B) \)

**Theorem 3.** If a function \( f(z) \in \Omega_p \) defined by (1.2) is in the class \( A_{\alpha,p}^n (a,b,c;\alpha,A,B) \) then

(i) \( f(z) \) is meromorphically \( p \)-valent starlike of order \( \gamma \) \((0 \leq \gamma < p)\) in \( |z| < r_1 \), where

\[
r_1 = \inf_{k \geq 0} \left\{ \frac{Q_{k+1}^n (a,b;c)}{(k+p)(p-\gamma)M_k (\alpha,A,B,P)} \right\} \left( \frac{1}{k+2p} \right).
\]

(2.12)

(ii) \( f(z) \) is meromorphically \( p \)-valent convex of order \( \gamma \) \((0 \leq \gamma < p)\) in \( |z| < r_2 \), where

\[
r_2 = \inf_{k \geq 0} \left\{ \frac{Q_{k+1}^n (a,b;c)}{(k+p+\gamma)(B-A)(p-\alpha)} \right\} \left( \frac{1}{k+2p} \right).
\]

(2.13)

The result are sharp.

**Proof.** From (1.2), we easily get
Thus, we have the desired inequity:

\[
\left| \frac{z f'(z) + p}{f(z)} - p + 2\gamma \right| \leq \frac{\sum_{k=0}^{n} (k + p) a_{k+p} |z|^{k+2p}}{2(\gamma + p) - \sum_{k=0}^{n} (k + 2\gamma) a_{k+p} |z|^{k+2p}}.
\]

(2.14)

If

\[
\sum_{k=0}^{n} (k + p + \gamma) a_{k+p} |z|^{k+2p} \leq 1.
\]

(2.15)

Hence, by Theorem 1, (2.15) will be true if

\[
\frac{(k + p + \gamma)}{(p - \gamma)} |z|^{k+2p} \leq \frac{(k + p) Q_{k+\gamma}^n(a,b;c) M_k(\alpha,A,B,P)}{P(B - A)(P - \alpha)} (k \geq 0, p \in N)
\]

(2.16)

The inequality (2.16) leads us immediately to |z| < r₁, where r₁ is given by (2.12).

(ii) In order to prove the second assertion of the Theorem we find from (1.2) that

\[
\left| 1 + z f''(z) + p \right| \leq \frac{\sum_{k=0}^{\infty} (k + p)(k + 2p) a_{k+p} |z|^{k+2p}}{2p(p - \gamma) - \sum_{k=0}^{\infty} (k + 2\gamma) a_{k+p} |z|^{k+2p}} .
\]

Thus we have the desired inequity:

\[
\left| 1 + z f''(z) + p \right| \leq \frac{\sum_{k=0}^{\infty} (k + p)(k + \gamma) a_{k+p} |z|^{k+2p}}{p(p - \gamma)} \leq 1.
\]

(2.17)

If

\[
\sum_{k=0}^{\infty} (k + p + \gamma) a_{k+p} |z|^{k+2p} \leq 1.
\]

(2.18)

Hence, by Theorem 1, (2.18) will be true if

\[
\frac{(k + p)(k + p + \gamma)}{p(p - \gamma)} |z|^{k+2p} \leq \frac{(k + p) Q_{k+\gamma}^n(a,b;c) M_k(\alpha,A,B,P)}{P(B - A)(P - \alpha)} (k \geq 0, p \in N),
\]

(2.19)

the inequality (2.19) leads us immediately to |z| < r₂ where r₂ is given by (2.13).

Each of these result is sharp for the function f(z) given by (2.9).

Next we prove closure Theorems for the class $A_{\lambda,p}^n(a,b,c;\alpha,A,B)$

**Theorem 4.** Let

\[
f_{-1} = \frac{1}{z^p}
\]

and

\[
f_{p+k}(z) = \frac{1}{z^p} - \frac{p(B - A)(P - \alpha)}{(P + K) Q_{k+\gamma}^n(a,b;c) M_k(\alpha,A,B,P)} Z^{p+k} (k \geq 0; p \in N; n \in N_0)
\]

(2.20)

Then f(z) in the class $A_{\lambda,p}^n(a,b,c;\alpha,A,B)$ if and only if it can expressed in the form
\[ f(z) = \sum_{k=1}^{\infty} \mu_{p+k} f_{p+k}(z), \]

where
\[ \mu_{p+k} \geq 0 \text{ and } \sum_{k=1}^{\infty} \mu_{p+k} = 1. \]

**Proof.** Let \( f(z) = \sum_{k=1}^{\infty} \mu_{p+k} f_{p+k}(z) \), where \( \mu_{p+k} \geq 0 \) and \( \sum_{k=1}^{\infty} \mu_{p+k} = 1 \).

Then
\[ f(z) = \sum_{k=1}^{\infty} \mu_{p+k} f_{p+k}(z), \]

\[ f(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} M_{p+k} \frac{P(B-A)(P-\alpha)}{(P+K)Q^n_{k,j}(a,b;c)M_k(\alpha,A,B,P)} z^{p+k}. \]

then
\[ \sum_{k=0}^{\infty} \mu_{p+k} \frac{p(B-A)(P-\alpha)}{(P+K)Q^n_{k,j}(a,b;c)M_k(\alpha,A,B,P)} (p+k)Q^n_{k,j}(a,b;c)M_k(\alpha,A,B,P) = 1 - \mu_{p-1} \leq 1, \]

which shows that \( f(z) \in A_{\lambda,p}^n(a,b,c;\alpha,A,B) \).

Conversely, Let \( f(z) \in A_{\lambda,p}^n(a,b,c;\alpha,A,B) \) then
\[ a_{k+p} \leq \frac{p(B-A)(P-\alpha)}{(P+K)Q^n_{k,j}(a,b;c)M_k(\alpha,A,B,P)} a_{k+p}. \]

Setting
\[ \mu_{p+k} \leq \frac{p(B-A)(P-\alpha)}{(P+K)Q^n_{k,j}(a,b;c)M_k(\alpha,A,B,P)} a_{k+p}, \]

and
\[ \mu_{p-1} = 1 - \sum_{k=0}^{\infty} \mu_{p+k}. \]

it follows that \( f(z) = \sum_{k=1}^{\infty} \mu_{p+k} f_{p+k}(z) \). This completes the proof of Theorem.

**Theorem 5.** The class \( A_{\lambda,p}^n(a,b,c;\alpha,A,B) \) is closed under convex linear combinations.

**Proof.** Let each of the functions
\[ f_j(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} a_{k+j} z^{p+k} (a_{k+j} \geq 0; j = 1,2) \]
be in the class \( A_{\lambda,p}^n(a,b,c;\alpha,A,B) \). It sufficient to show that the function \( h(z) \) defined by
\[ h(z) = (1-t)f_1(z) + tf_2(z) \in A_{\lambda,p}^n(a,b,c;\alpha,A,B) \text{ (} 0 \leq t \leq 1 \),

is also in the class \( A_{\lambda,p}^n(a,b,c;\alpha,A,B) \) since
\[ h(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} [(1-t)a_{k+1} + ta_{k+2}] z^{p+k} \text{ (} 0 \leq t \leq 1 \).

With the aid of Theorem 1, we have
\[ \sum_{k=0}^{\infty} (p+k)Q^n_{k,j}(a,b;c) M_k(\alpha,A,B,P) [(1-t)a_{k+1} + ta_{k+2}] \]
$$= (1-t) \sum_{k=0}^{\infty} \left( p + k \right) \mathcal{Q}_{k}^{n}(a,b;c) \left( M_{k} (\alpha, A, B, P) \alpha_{p+k,1} ight)$$

$$+ t \sum_{k=0}^{\infty} \left( p + k \right) \mathcal{Q}_{k}^{n}(a,b;c) \left( M_{k} (\alpha, A, B, P) \alpha_{p+k,2} \right)$$

$$\leq (1-t) p(B - A)(P - \alpha) + tp(B - A)(P - \alpha) = p(B - A)(P - \alpha),$$

which shows that $h(z) \in A_{\alpha, p}^{n}(a,b,c;\alpha, A, B)$.

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