Region of Variability of a Subclass of Starlike Univalent Functions

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Abstract: Let $R$ denote the class of all analytic univalent functions $f(z)$ in the unit disk $\Delta$ with $f(0) = f'(0) = 1$ and $zf'(z)/f(z)$ is starlike. For any fixed $z_0$ in the unit disk and $\lambda \in \Delta$, we determine the region of variability $V(z_0, \lambda)$ for

$$\log \frac{f(z_0)}{f(z)}$$

when $f$ ranges over the class $R(\lambda) = \{ f \in R : f'(0) = 2\lambda + 1 \}$.

Keywords: Analytic functions, Schwarz lemma, Variability region.

Mathematics Subject Classification: Primary 30C45

INTRODUCTION

Denote by $\mathcal{H}(\Delta)$ the class of analytic functions in the unit disk $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$. The set $\mathcal{H}(\Delta)$ may be thought as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\Delta$. Let $\mathcal{A}$ denote the set of all functions in $\mathcal{H}(\Delta)$ such that $f(0) = f'(0) = 1$ and $S$ be the class of all functions in $\mathcal{A}$ that are univalent. Several researchers have studied the region of variability problems of $f$ at a specified point inside the unit disk for several subclasses of $S$. In [2], the problem of determining the region of values of $\log \frac{f(z)}{f(z_0)}$ for a fixed $z_0 \in \Delta$ as $f$ ranges over the class $S^*$ of starlike functions is given. Duren in his paper [3] discusses the region of variability of $f(z_0)$ for $f \in S$ and $g(z_0)$ for $g \in S_0 = \{ f \in \mathcal{A} : f(z) \neq 0 \text{ in } \Delta, f'(0) = 1 \}$. Bhowmik determined the region of variability for concave univalent functions [1]. In [5, 6, 7, 8, 9, 10], S. Ponnusamy et al. had obtained the region of variability for several standard subclasses of $S$. H. Yanagihara had discussed the region of variability for functions with bounded derivatives, convex functions and families of convex functions in [11, 12, 13].

In this paper, we define a new subclass of univalent analytic functions $f$ satisfying certain normalization condition and determine the region of variability of $\log \frac{f(z_0)}{f(z)}$.

Let $R$ denote the class of all analytic univalent functions $f(z)$ in the unit disk $\Delta$ with $f(0) = f'(0) = 1$ and $\Re \frac{zf'(z)}{f(z)} > 0, z \in \Delta$ where $F(z) = \frac{zf'(z)}{f(z)}$. Let

$$P_f(z) = \frac{zf'(z)}{f(z)} = 1 + \frac{zf'(z) - zf'(z)}{f(z)}$$

(1)

Clearly $P_f(0) = 1$. For $f \in R$, we denote by $\log \frac{f(z_0)}{f(z)}$ the single valued branch of logarithm of $\frac{f'(z_0)}{f(z)}$. Herglotz representation for analytic functions with positive real part in $\Delta$ shows that if $f \in R$ then there exists a unique positive measure $\mu$ on $(-\pi, \pi]$ such that

$$1 + \frac{zf'(z)}{f(z)} = \int_{-\pi}^{\pi} 1 + z e^{-it} \frac{zf'(z)}{f(z)} dt, \ z \in \Delta$$

A computation gives

$$\log \frac{f'(z_0)}{f(z_0)} = 2 \int_{-\pi}^{\pi} \log \frac{1}{1 - ze^{-it}} d\mu(t)$$
For each fixed \( z_0 \in \Delta \), the region of variability \( V(z_0) = \{ \log \frac{f(z_0)}{f(z)} : f \in \mathcal{R} \} \) coincides with the set \(-2 \log(1 - z) : |z| \leq |z_0| \). Let \( \mathcal{B}_0 \) denote the class of analytic functions \( \omega \) in \( \Delta \) such that \( |\omega(z)| \leq 1 \) in \( \Delta \) and \( \omega(0) = 0 \). Then for each \( f \in \mathcal{R} \), there exist \( \omega_f \in \mathcal{B}_0 \) of the form
\[
\omega_f(z) = \frac{P_f(z)}{P_f(z) + 1}, \quad z \in \Delta
\]
and conversely. Clearly
\[
P'_f(0) = 2\omega'_f(0) = f''(0) - 1
\]
If \( f \in \mathcal{R} \) then a simple application of Schwarz lemma shows that
\[
|P'_f(0)| = |f''(0) - 1| \leq 2
\]
Since \( |\omega'_f(0)| \leq 1 \). This implies that \( f''(0) = 2\lambda + 1 \) for some \( \lambda \in \overline{\Delta} = \{ z \in \mathbb{C} : |z| \leq 1 \} \).
For \( \omega \in \mathcal{B}_0 \) define \( g : \Delta \rightarrow \overline{\Delta} \) by
\[
g(z) = \begin{cases} \frac{\omega_f(z)}{1 - \frac{1}{2} \omega_f(z)} & \text{if } |\lambda| < 1 \\ 0 & \text{if } |\lambda| = 1 \end{cases}
\]
Then
\[
g'(0) = \begin{cases} \frac{\omega''_f(0)}{2(1 - |\lambda|^2)} & \text{if } |\lambda| < 1 \\ 0 & \text{if } |\lambda| = 1 \end{cases}
\]
For \( |\lambda| < 1 \),
\[
|g'(0)| < 1 \iff \frac{|\omega''_f(0)|}{2(1 - |\lambda|^2)} \leq 1
\]
\[
\iff \frac{P''_f(0) - 2\lambda - 2\lambda^2}{2(1 - |\lambda|^2)} \leq 1
\]
\[
\iff P''_f(0) = 2\alpha(1 - |\lambda|^2) + 2\lambda(\lambda + 1)
\]
for some \( \alpha \in \overline{\Delta} \). For \( \lambda \in \overline{\Delta} \) and a fixed \( z_0 \in \Delta \), introduce
\[
\mathcal{R}(\lambda) = \{ f \in \mathcal{R} : f''(0) = 2\lambda + 1 \}
\]
and
\[
V(z_0, \lambda) = \{ \log \frac{f(z_0)}{f(z)} : f \in \mathcal{R}(\lambda) \}
\]
The aim of this paper is to determine the region of variability \( V(z_0, \lambda) \) of \( \log \frac{f(z_0)}{f(z)} \) when \( f \) ranges over the class \( \mathcal{R}(\lambda) \).

**Basic Properties of \( V(z_0, \lambda) \) and the Main Result**

For a positive integer \( p \), let
\[
(S^*)^p = \{ f = f_0^p : f \in S^* \}
\]
We now recall the following result from [12] to prove our main theorem.

**Lemma 2.1.** Let \( f \) be an analytic function in \( \Delta \) with \( f(z) = z^p + \ldots \). If
\[
\text{Re} \left( 1 + \frac{zf'(|z|)}{f'(|z|)} \right) > 0, z \in \Delta \text{ then } f \in (S^*)^p.
\]

**Theorem 2.1.** We have
(i) \( V(z_0, \lambda) \) is a compact subset of \( \mathbb{C} \).
(ii) \( V(z_0, \lambda) \) is a convex subset of \( \mathbb{C} \).
(iii) For \( |\lambda| = 1 \) or \( z_0 = 0 \), \( V(z_0, \lambda) = \{-2\log(1 - \lambda z_0)\} \)
(iv) For \( |\lambda| < 1 \) or \( z_0 \in \Delta - \{0\} \), \( V(z_0, \lambda) \) has \( -2\log(1 - \lambda z_0) \) as an interior point.
(v) \( V(e^{i\theta} z_0, \lambda) = V(z_0, e^{i\theta} \lambda) \) for \( \theta \in \mathbb{R} \), the set of real numbers.

**Proof:**
(i) Since \( \mathcal{R}(\lambda) \) is a compact subset of \( \mathcal{H}(\Delta) \), it follows that \( V(z_0, \lambda) \) is a compact subset of \( \mathbb{C} \).

(ii) Let \( f_1, f_2 \in \mathcal{R}(\lambda) \). Then for \( 0 \leq t \leq 1 \), the function

Available Online: [http://saspjournals.com/sjpms](http://saspjournals.com/sjpms)
Since \( \log \frac{f'(z_0)}{f(z_0)} = (1-t) \log \frac{f'(z_0)}{f(z_0)} + t \log \frac{f'(z_0)}{f_2(z_0)} \) \( V(z_0, \lambda) \) is a convex subset of \( \mathbb{C} \).

(iii) If \( z_0 = 0 \) then the result holds trivially. If \(|\lambda| = 1\) then \(|\omega'(0)| = 1\).

By Schwarz lemma, \( \omega(z) = \lambda z \) which implies

\[
P_f(z) = \frac{1 + \lambda z}{1 + \bar{\lambda} z}
\]

A computation gives

\[
\log \frac{f'(z)}{f(z)} = -2 \log(1 - \lambda z)
\]

which implies \( V(z_0, \lambda) = \{-2 \log(1 - \lambda z_0)\} \).

(iv) For \( \lambda \in \Delta, z_0 \in \Delta - \{0\} \) and \( a \in \Delta \) we define

\[
\delta(z, \lambda) = \frac{z + \lambda}{1 + \bar{\lambda} z}
\]

\[
F_{a, \lambda}(z) = \exp \int_0^z \left[ \exp \int_0^{\zeta_2} \frac{2\delta(a(\zeta, \lambda))}{1 - \zeta_1 \delta(a(\zeta, \lambda))} d\zeta_1 \right] d\zeta_2
\]

We prove that \( F_{a, \lambda} \) satisfying (5) belong to the class \( \mathcal{R}(\lambda) \). Note that

\[
1 + \frac{z F'_{a, \lambda}(z)}{F_{a, \lambda}(z)} - \frac{z F'_{a, \lambda}(z)}{F_{a, \lambda}(z)} = \frac{1 + z \delta(a z, \lambda)}{1 - z \delta(a z, \lambda)}
\]

Since \( \delta(az, \lambda) \) lies in the unit disk \( \Delta, F_{a, \lambda} \in \mathcal{R}(\lambda) \). Also note that

\[
\omega_{F_{a, \lambda}}(z) = z \delta(az, \lambda)
\]

The mapping \( \Delta \ni a \rightarrow \log \frac{f'(z_0)}{f(z_0)} \) is a non-constant analytic function of \( a \) for each fixed \( z_0 \in \Delta - \{0\} \) and \( \lambda \in \Delta \). To prove this we put

\[
h(z) = \frac{z}{1 - |z|^2} \int_0^z \frac{\zeta_2}{1 - \zeta_2} d\zeta_2 = \frac{z^2 + \ldots}{z}
\]

from which it is easy to see that

\[
\text{Re} \left\{ 1 + \frac{h(z)}{h(z)} \right\} > 0
\]

By lemma (2.1), there is a function \( h_0 \in S^* \) such that \( h = h_0^2 \). Since \( h_0 \) is univalent and \( h(0) = 0 \), we get \( h(z_0) \neq 0 \) for \( z_0 \in \Delta - \{0\} \).

Thus the map \( a \rightarrow \log \frac{F'_{a, \lambda}(z_0)}{F_{a, \lambda}(z_0)} = \int_0^{z_0} \frac{2 \delta(a(\zeta, \lambda))}{1 - (1 - |\zeta|^2)} d\zeta \) is a non-constant analytic function of \( a \) and hence is an open mapping. Thus \( V(z_0, \lambda) \) contains the open set \( \left\{ \log \frac{F'_{a, \lambda}(z_0)}{F_{a, \lambda}(z_0)} : |a| < 1 \right\} \).

In particular \( \log \frac{F_{a, \lambda}(z_0)}{F_{a, \lambda}(z_0)} = -2 \log(1 - \lambda z_0) \) is an interior point of

\[
\left\{ \log \frac{F'_{a, \lambda}(z_0)}{F_{a, \lambda}(z_0)} : |a| < 1 \right\} \subset V(z_0, \lambda)
\]

Since \( V(z_0, \lambda) \) is a compact subset of \( \mathbb{C} \) and has non-empty interior, the boundary \( \partial V(z_0, \lambda) \) is a Jordan curve and \( V(z_0, \lambda) \) is the union of \( \partial V(z_0, \lambda) \) and its interior domain.

(v) This follows from the fact that \( e^{-i\theta} f(e^{i\theta} z) \in \mathcal{R}(\lambda) \) if and only if \( f \in \mathcal{R} \).

We now state our main result and the proof will be presented in Section 3.

**Theorem 2.2.** For \( \lambda \in \Delta \) and \( z_0 \in \Delta - \{0\} \), the boundary \( \partial V(z_0, \lambda) \) is the Jordan curve given by

\[
(-\pi, \pi) \ni \theta \rightarrow \log \frac{F'_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z_0)} = \int_{z_0}^{z_0} 2 \delta(e^{i\theta} \zeta, \lambda) d\zeta.
\]

If \( \log \frac{f'(z_0)}{f(z_0)} = \log \frac{F'_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z_0)} \) for some \( f \in \mathcal{R} \) then \( f(z) = F_{e^{i\theta}, \lambda}(z) \).
Theorem 3.1. For $f \in \mathcal{R}(\lambda), \lambda \in \Delta$ we have

$$
\left\| \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) - c(z, \lambda) \right\| \leq r(z, \lambda), \quad z \in \Delta
$$

(7)

where

$$
c(z, \lambda) = \frac{2\nu((1-|z|^2) + x|z|^2)}{(1-|z|^2)(1-2\nu Re(zx) + |x|^2)}$$

and

$$
\nu = \frac{2|z|}{(1-|z|^2)(1-2|z|^2) + |x|^2}
$$

For each $z \in \Delta - \{z_0\}$, equality holds if and only if $f = F_e^{i\theta, \lambda}$.

Proof. Let $f \in \mathcal{R}(\lambda)$. Then there exists a $\omega_f(z) \in \mathcal{B}_0$ satisfying $\omega_f(z) = \frac{P_f(z)}{P_f(z) + 1}$ where $P_f(z)$ is defined by (1). Since $\omega_f(0) = \lambda$, by Schwarz lemma,

$$
\left\| \frac{\omega_f(z)}{1 - \omega_f(z)} \right\| \leq |z|, \quad z \in \Delta
$$

(8)

which by definition of $P_f$ is equivalent to

$$
\left\| \frac{f''(z)}{f'(z)} - A(z, \lambda) \right\| \leq |z||r(z, \lambda)|
$$

(9)

where $A(z, \lambda) = \frac{2\lambda}{1-\lambda}$, $B(z, \lambda) = \frac{2}{z-\lambda}$, $r(z, \lambda) = \frac{z-\lambda}{1-\lambda}$.

A simple calculation shows that (9) is equivalent to

$$
\left\| \frac{f''(z)}{f'(z)} - A(z, \lambda) + \frac{|z|^2|r(z, \lambda)|^2 B(z, \lambda)}{1 - |z|^2|r(z, \lambda)|^2} \right\| \leq \frac{|z||r(z, \lambda)| |A(z, \lambda) + \theta(z, \lambda)||}{1 - |z|^2|r(z, \lambda)|^2}
$$

(11)

A computation gives

$$
1 - |z|^2|r(z, \lambda)|^2 = \frac{(1-|z|^2)(1-2|z|^2 + |x|^2)}{|1-\lambda|^2}
$$

(12)

$$
A(z, \lambda) + B(z, \lambda) = \frac{2(1-|z|^2)}{(1-\lambda)(1-\lambda)}
$$

(13)

$$
A(z, \lambda) + |z|^2|r(z, \lambda)|^2 B(z, \lambda) = \frac{2|z|(1-|z|^2 + x|z|^2)}{|1-\lambda|^2}
$$

(14)

Using the above equations we find that

$$
\frac{A(z, \lambda) + |z|^2|r(z, \lambda)|^2 B(z, \lambda)}{1 - |z|^2|r(z, \lambda)|^2} = c(z, \lambda)
$$

and

$$
\frac{|z||r(z, \lambda)| |A(z, \lambda) + \theta(z, \lambda)||}{1 - |z|^2|r(z, \lambda)|^2} = r(z, \lambda)
$$

The inequality in (7) follows from these equalities and (11). The equality occurs in (7) for any $z \in \Delta$ only when $f = F_e^{i\theta, \lambda}$ for some $\theta \in \mathbb{R}$. Conversely if the equality occurs for some $z \in \Delta - \{0\}$ in (7) then the equality must hold in (8). By Schwarz lemma there exists a $\theta \in \mathbb{R}$ such that $\omega_f(z) = z\delta(e^{i\theta}z, \lambda)$ for all $z \in \Delta$. This implies $f = F_e^{i\theta, \lambda}$.

When $\lambda = 0$ we have the following result.

Corollary 3.1. For $f \in \mathcal{R}(0)$

$$
\left\| \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right\| \leq \frac{2|z|}{1 - |z|^2}, \quad z \in \Delta
$$

(15)

For each $z \in \Delta - \{0\}$, equality holds if and only if $f = F_e^{i\theta, \lambda}$ for some $\theta \in \mathbb{R}$. In particular

$$
(1 - |z|^2) \left| \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right| \leq 2|z|
$$

Corollary 3.2. Let $y : z(t), \quad 0 \leq t \leq 1$ be a $C^1$-curve in $\Delta$ with $z(0) = 0$ and $z(1) = z_0$. Then

$$
V(z_0, \lambda) \subset \{ w \in \mathbb{C} : |w - C(\lambda, \gamma)| \leq R(\lambda, \gamma) \}
$$

where $C(\lambda, \gamma) = \int_0^1 C(z(t), \lambda)z'(t) dt$ and $R(\lambda, \gamma) = \int_0^1 r(z(t), \lambda)|z'(t)| dt$.
Proof. Let \( f \in \mathcal{R}(\lambda) \). Then by theorem 3.1,
\[
\left| \log \frac{f'(z_0)}{f(z_0)} - C(\lambda, \gamma) \right| = \left| \int_0^1 \left( \frac{f''(z)}{f'(z)} \right)' - c(z(t), \lambda) \right| z'(t) \, dt \\
\leq \int_0^1 r(z(t), \lambda) |z'(t)| \, dt \\
= R(\lambda, \gamma)
\]
Since \( \log \frac{f'(z_0)}{f(z_0)} \in \mathcal{V}(z_0, \lambda) \) was arbitrary, the result follows.
To prove our next result we need the following lemma [10].

**Lemma 3.1.** For \( \theta \in \mathbb{R}, \lambda \in \Delta \), the function
\[
G(z) = \int_0^z \frac{e^{i\theta \zeta}}{1 + (\lambda e^{i\theta} - \lambda - e^{i\theta} \zeta)^2} \, d\zeta , \quad z \in \Delta
\]
has a double zero at the origin and no zeros elsewhere in \( \Delta \). Furthermore, there exists a starlike univalent function \( G_0 \) in \( \Delta \) such that \( G = e^{i\theta} G_0^2 \), \( G_0(0) = G_0''(0) = 1 \).

**Theorem 3.2.** Let \( z_0 \in \Delta - \{0\} \). Then for \( \theta \in (-\pi, \pi] \) we have \( \log \frac{F_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z)} \in \partial V(z_0, \lambda) \). Further if \( \log \frac{f'(z_0)}{f(z_0)} = \log \frac{F_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z)} \) for some \( f \in \mathcal{R}(\lambda) \) and \( \theta \in (-\pi, \pi] \) then \( F_{e^{i\theta}, \lambda} \).

Proof. From (5)
\[
\frac{F'_{e^{i\theta}, \lambda}(z)}{F_{e^{i\theta}, \lambda}(z)} - c(z, \lambda) = \frac{2\delta(az, \lambda)}{1 - z\delta(az, \lambda)} = \frac{-2(az + \lambda)}{az^2 + (\lambda - \lambda a)z - 1} - 2az(\lambda^2 - 1)
\]
and hence we obtain
\[
\frac{F'_{e^{i\theta}, \lambda}(z)}{F_{e^{i\theta}, \lambda}(z)} - c(z, \lambda) = \frac{2(\lambda^2 - 1)(a(1 - \lambda z) - |z|^2(z - 1))}{(1 - |z|^2)(1 - 2Re(\lambda z) + |z|^2)(az^2 + (\lambda - \lambda a)z - 1)}
\]
Substituting \( a = e^{i\theta} \) we have
\[
\frac{F'_{a, \lambda}(z)}{F_{a, \lambda}(z)} - c(z, \lambda) = r(z, \lambda) \frac{ze^{i\theta} \left[ 1 + (\lambda e^{i\theta} - \lambda)z - e^{i\theta}z^2 \right]^2}{|z|^2 \left[ 1 + (\lambda e^{i\theta} - \lambda)z - e^{i\theta}z^2 \right]^2}
\]
Using above lemma,
\[
\frac{F_{a, \lambda}(z)}{F_{a, \lambda}(z)} - \frac{F'_{a, \lambda}(z)}{F_{a, \lambda}(z)} - c(z, \lambda) = r(z, \lambda) \frac{G'(z)}{G(z)}
\]
(15)
Since \( G_0 \) is starlike, for any \( z_0 \in \Delta - \{0\} \), the linear segment joining 0 and \( G(z_0) \) lies entirely in \( G_0(\Delta) \). Define \( \gamma_0 \) by
\[
\gamma_0 : z(t) = G_0^{-1}(tG_0(z_0)), \quad 0 \leq t \leq 1
\]
Since \( G(z(t)) = 2^{-1}e^{i\theta} G_0(z(t))^2 = 2^{-1}e^{i\theta} G_0(z(t))^2 = t^2 G(z_0) \), we have
\[
G'(z(t))z'(t) = 2G(z_0), \quad t \in [0, 1]
\]
Using (15) and (17) we have
\[
\log \frac{F'_{e^{i\theta}, \lambda}(z)}{F_{e^{i\theta}, \lambda}(z)} = C(\lambda, \gamma_0) = \int_0^1 \left( \frac{F'_{e^{i\theta}, \lambda}(z(t))}{F'_{e^{i\theta}, \lambda}(z(t))} - \frac{F'_{e^{i\theta}, \lambda}(z(t))}{F_{e^{i\theta}, \lambda}(z(t))} - c(z(t), \lambda) \right) z'(t) \, dt \\
= \int_0^1 r(z(t), \lambda) \frac{G'(z(t))z(t)}{G(z(t))} \, dt \\
= \frac{G(z_0)}{G(0)} R(\lambda, \gamma_0)
\]
This implies that \( \log \frac{F'_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z_0)} \in \partial V(z_0, \lambda) \subset \Delta(C(\lambda, \gamma_0), R(\lambda, \gamma_0)) \). By Corollary (3.2) we have \( \log \frac{F'_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z_0)} \in V(z_0, \lambda) \subset \Delta(C(\lambda, \gamma_0), R(\lambda, \gamma_0)) \).
which implies \( \log \frac{F_{e^{i\theta}, \lambda}(z)}{F_{e^{i\theta}, \lambda}(z_0)} \in \partial V(z_0, \lambda) \). For uniqueness, suppose

\[
\log \frac{f'(z_0)}{f(z_0)} = \log \frac{F'_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z_0)}
\]

for some \( f \in \mathcal{R}(\lambda) \). Introduce

\[
h(t) = \frac{G(z_0)}{|G(z(t)|} \left( \frac{f''(z(t))}{f'(z(t))} - \frac{f'(z(t))}{f(z(t))} - c(z(t), \lambda) \right) z'(t) dt
\]

where \( \gamma_0 : z(t), 0 \leq t \leq 1 \), then \( h(t) \) is continuous function on \([0,1]\) and satisfies

\[
|h(t)| \leq |r(z(t), \lambda)| |z(t)'|, \quad 0 \leq t \leq 1
\]

Also we have

\[
\int_0^1 \text{Re}(h(t)) dt = \int_0^1 \text{Re} \left( \frac{G(z_0)}{|G(z(t)|} \left( \frac{f''(z(t))}{f'(z(t))} - \frac{f'(z(t))}{f(z(t))} - c(z(t), \lambda) \right) z'(t) \right) dt
\]

\[
= \text{Re} \left( \frac{G(z_0)}{|G(z(t)|} \left( \log \frac{f'(z(t))}{f(z(t))} - C(\lambda, \gamma_0) \right) \right)
\]

\[
= \int_0^1 \tau(z(t), \lambda)|z'(t)| dt
\]

which implies \( h(t) = \tau(z(t), \lambda)|z'(t)| \) for all \( t \in [0,1] \). From (15) and (17) it follows that

\[
\frac{f''}{f'} - \frac{f'}{f} = \frac{F''_{e^{i\theta}, \lambda}(z(t))}{F'_{e^{i\theta}, \lambda}(z(t))} - \frac{F'_{e^{i\theta}, \lambda}(z(t))}{F_{e^{i\theta}, \lambda}}
\]

on the curve \( \gamma_0 \). By normalization \( f = F_{e^{i\theta}, \lambda} \).

**Proof of theorem 2.2**

We need to prove that the closed curve

\[
(-\pi, \pi) \ni \theta \rightarrow \log \frac{F'_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z_0)}
\]

is simple. Suppose that \( \log \frac{F'_{e^{i\theta_1}, \lambda}(z_0)}{F_{e^{i\theta_1}, \lambda}(z_0)} = \log \frac{F'_{e^{i\theta_2}, \lambda}(z_0)}{F_{e^{i\theta_2}, \lambda}(z_0)} \)

for some \( \theta_1, \theta_2 \in (-\pi, \pi), \theta_1 \neq \theta_2 \). By Theorem (3.2), \( F_{e^{i\theta_1}, \lambda} = F_{e^{i\theta_2}, \lambda} \)

From (6) this gives a contradiction that

\[
e^{i\theta_1} = \tau \left( \frac{\omega_{F_{e^{i\theta_1}, \lambda}}}{z}, \lambda \right) = \tau \left( \frac{\omega_{F_{e^{i\theta_2}, \lambda}}}{z}, \lambda \right) = e^{i\theta_2}
\]

which implies the curve is simple. Since \( V(z_0, \lambda) \) is compact convex subset of \( C \) and has non-empty interior, the boundary \( \partial V(z_0, \lambda) \) is a simple closed curve. By Theorem 3.1, the curve contains the curve \( (-\pi, \pi) \ni \theta \rightarrow \log \frac{F'_{e^{i\theta}, \lambda}(z_0)}{F_{e^{i\theta}, \lambda}(z_0)} \). Since a simple closed curve cannot contain any simple closed curve other than itself, the result follows immediately.

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