Discussion on a Kind of Sequence Limit
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Abstract: In this paper, we give four theorems and proved them. According to these four theorems, we deduce the solver method for the limit of a class of sequence \( \{x_n\} \) by recursive relation \( x_n = f(x_{n-1}) \).

Keywords: limit of a sequence, recursion formula, mathematical induction limit of sequence

PROBLEM POSING
For a special class of infinite series \( (n=1,2,L) \), If sequence of \( \{x_n\} \) satisfies the recursive formula of \( x_n = f(x_{n-1}) \), we can discuss the limit of \( \{x_n\} \) \((n=1,2,L)\) according to the property of \( f'(x) \) [1]. So we give four theorems as follows.

FOUR THEOREMS
Theorem 1
If in \([a,b]\) equation \( x = f(x) \) has a unique root \( \xi \), and
\[ |f'(x)| \leq q < 1, \ x_0 \] is any real number in \([\xi - x_0, \xi + x_0]\), then sequence :
\[ x_1 = f(x_0), x_2 = f(x_1), ... x_n = f(x_{n-1}) \]
convergence to \( \xi \).

Proof
According to \( x_1 - f(x_0) = \xi - f(\xi) \) and Lagrange Mean Value Theorem[2]
\[ |x_1 - x_0| = |x_1 - x_0| |f'(c)| \leq |f'(c)| < q < |\xi - x_0| \ (\xi < c < x_0) \],
so \( x_1 \) is closer to \( \xi \) than \( x_0 \), moreover, as \( |x_1 - x_0| \leq \min\{(|\xi - a|, |b - \xi|)\} \), so \( x_1 \in [a,b] \).

By mathematical induction[3], we can deduce \( x_n \in [a,b] \) \((n=1,2,L)\). According to
\[ x_{n+1} - f(x_n) = x_n - f(x_{n-1}) \]
and Lagrange Mean Value Theorem :
\[ |x_{n+1} - x_n| = |x_{n+1} - x_{n-1}| |f'(c_{n})| \].

Since \( x_n \in [a,b] \), \( c_n \) is between \( x_n \) and \( x_{n-1} \), so \( c_n \in [a,b] \), \( |f'(c_n)| \leq q < 1 \). Therefore
\[ |x_{n+1} - x_n| \leq |x_n - x_{n-1}| q \].

And then
\[ |x_{n+1} - x_n| \leq q^n |x_1 - x_0|, |x_{n+p} - x_n| \leq |x_1 - x_0| \leq \frac{q^n}{1-q} |x_1 - x_0| \].

When \( n \to \infty, q^n \to 0 \), so \( \lim_{n \to \infty} x_n \) is extant. By the continuity of \( f(x) \), \( \lim_{n \to \infty} x_n = f(\lim_{n \to \infty} x_{n-1}) \), and the uniqueness of \( \xi \) in \([a,b]\), \( \lim_{n \to \infty} x_n = \xi \).
Theorem 2 If \( f = f(x) \) has real root \( \xi_i \) (\( i = 1, 2, 3, \ldots, m \)) \( f(x) \) has derivative in every point. And \( |f'(\xi_i)| > 1 \), for any real number \( x_0 \), the follow sequence is diverging:

\[
x_i = f(x_{i-1}), x_2 = f(x_1), \ldots, x_n = f(x_{n-1}), \ldots \text{ (if } i \neq j \text{ then } x_i \neq x_j)\]

**Proof** If \( x_n = f(x_{n-1}) \), sequence \( \{ x_n \} \) must converge to some real number \( \xi_N \) (\( 1 \leq N \leq m \)).

By \( x_n - f(x_{n-1}) = \xi_n - f(\xi_n) \) and Lagrange Mean Value Theorem, we can derive \( |\xi_N - x_n| \Rightarrow |f'(\xi_n)| \) (\( n \rightarrow \infty \)).

Since \( c \) is between \( \xi_N \) and \( x_{n-1} \), \( x_n \rightarrow \xi_N \) (\( n \rightarrow \infty \)), we can know \( \xi_N \) is tenable.

**Theorem 3** If in \([a, b]\) equation \( x = f(x) \) has a unique root \( \xi \), \( f'(x) < 0 \), and \( f(a) \in [a, b] \). \( f(b) \in [a, b] \), \( x_0 \in [a, b] \), then sequence:

\[
x_1 = f(x_0), x_2 = f(x_1), x_{2m+1} = f(x_{2m}), x_{2m+2} = f(x_{2m+1})\]

and \( x_2 = f(x_1), x_4 = f(x_3), x_{2m} = f(x_{2m-1}) \) are both convergent.

**Proof** We might as well let \( x_0 = b \) (\( x_0 \) is any value in \([a, b]\), the proof is same as this) . According to \( x_1 - f(b) = \xi - f(\xi) \) and \( f'(x) < 0 \), \( \xi < b \), we can derive \( x_1 < \xi \). By \( x_2 - f(x_1) = \xi - f(\xi) \) and \( f'(x) < 0 \), \( x_2 < \xi \), we can know \( x_2 > \xi \). \( x_{2m} > \xi > x_{2m+1} \) \( (m = 0, 1, 2, \ldots) \) can be proved by mathematical induction.

Set \( x_1 = f(a) \), by \( x_2 - f(x_1) = x_1 - f(a) \) and \( f'(x) < 0 \), \( x_1 \geq a \), we can deduce \( x_2 \leq x_1 \). we have known \( x_1 \leq b \) \( (x_1 \in [a, b]) \), so \( x_2 \leq b \).

As \( x_3 - f(x_2) = x_2 - f(b) \) and \( f'(x) < 0 \), \( x_2 \leq b \), then \( x_3 \leq x_2 \), By \( x_2 - f(x_1) = x_2 - f(x_3) \) and \( f'(x) < 0 \), \( x_1 \leq x_3 \), we can know \( x_4 \leq x_2 \).

So \( x_{2m-1} \leq x_{2m+1} \leq x_{2m} \leq x_{2m-2} \) \( (m = 0, 1, 2, \ldots) \) might be proved by mathematical induction. In conclusion, then

\[
x_{2m-1} \leq x_{2m+1} \leq x_{2m} \leq x_{2m-2} \quad (m = 0, 1, 2, \ldots)[3],
\]

so we can see \( \lim_{m \rightarrow \infty} x_{2m+1} \) and \( \lim_{m \rightarrow \infty} x_{2m} \) are both existent[4].

**Theorem 4** If in \([a, b]\) equation \( x = f(x) \) has a unique root \( \xi \), and \( 0 < f'(x) < 1 \), \( x_0 \in [a, b] \), then sequence:

\[
x_1 = f(x_0), x_2 = f(x_1), x_n = f(x_{n-1})\]

converge to \( \xi \).
Proof We might as well let \( x_0 = b \). According to \( 0 < f'(x) < 1 \), we can derive \( 1 - f'(x) > 0 \). So \( x - f(x) \) is monotone increasing, so that \( b - f(b) > 0 \), \( a - f(a) < 0 \). By \( f'(x) > 0 \), we can know \( a - f(b) < a - f(a) < 0 \). So monotone continuous function \( x - f(b) \) in the two endpoints of \([a, b]\) has opposite signs. So that \( x_1 = f(b) \in [a, b] \). As \( x_1 - f(b) = \xi - f(\xi) \) and \( f'(x) > 0 \), \( b > \xi \), then \( x_1 > \xi \). So \( x_n > \xi \) \((n = 0, 1, 2, L)\) might be proved by mathematical induction. By \( x_1 - f(b) = x_2 - f(x_1) \), \( f'(x) > 0 \), \( b > x_1 \), we can know \( x_1 > x_2 \). By mathematical induction we can prove \( x_n > x_{n+1} \) \((n = 0, 1, 2, L)\), so that \( \xi < x_{n+1} < x_n \) \((n = 0, 1, 2, L)\). So \( \lim_{n \to \infty} x_n \) is existent. By \( x_n = f(x_{n-1}) \) we can infer

\[
\lim_{n \to \infty} x_n = f(\lim_{n \to \infty} x_{n-1})
\]

Since \( \xi \) is unique in \([a, b]\), we can infer \( \lim_{n \to \infty} x_n = \xi \).

APPLICATION EXAMPLE

Example 1 Sequence \( \{ x_n \} \) is as follows.

\[
x_1 = \sqrt{2}, x_2 = \sqrt{2 + \sqrt{2}, L} , x_n = \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}, L}}
\]

Solving \( \lim_{n \to \infty} x_n \)[5].

Solving \( x = \sqrt{x + 2} \), the solution is 2. In \([0,4]\), the equation has a unique solution 2, and

\[
| (\sqrt{x + 2})' | = \left| \frac{1}{2\sqrt{x + 2}} \right| < \frac{1}{2} < 1 , \text{let} \ x_0 = 0 , \text{it meets qualifications of Theorem 1, so} \ \lim_{n \to \infty} x_n = 2 .
\]

Example 2 Solving \( \lim_{n \to \infty} (a + a^2 + \ldots + a^n) \) \((|a| < 1)\).

Solving \( x = ax + a \), the solution is \( x = \frac{a}{1-a} \). In \([0, \frac{2a}{1-a}] \) it has a unique solution \( \frac{a}{1-a} \) (we might as well set \( a > 0 \)) , and \( | (ax + a) ' | = |a| < 1 \). Let \( x_0 = 0 \), we can know it meets qualifications of Theorem 1, so

\[
\lim_{n \to \infty} (a + a^2 + \ldots + a^n) = \frac{a}{1-a} . \text{If} \ a \leq 0 (|a| < 1) , \ \lim_{n \to \infty} (a + a^2 + \ldots + a^n) = \frac{a}{1-a} .
\]

When \( |a| < 1 \),

\[
\lim_{n \to \infty} (a + a^2 + \ldots + a^n) = \frac{a}{1-a} .
\]

When \( |a| > 1 \), we can infer \( | (ax + a) ' | = |a| > 1 \). Using theorem 2 we can know sequence:

\[
x_i = a, x_2 = a + a^2 , \ldots , x_n = a + a^2 + \ldots + a^n , \ldots \ (i \neq j \parallel j , \ x_i \neq x_j) \text{ is divergent.}
\]

CONCLUSION

In general, through the four theorems we can deduce the solver method for the limit of a class of sequence \( \{ x_n \} \) by recursive relation \( x_n = f(x_{n-1}) \).
First, solving \( x = f(x) \) we got all the real number solutions \( \xi_i \ (i = 1, 2, L, m) \). Sequence \( \{ x_n \} \) converge to and can only converge to some \( \xi_i \). And according to the features of sequence \( \{ x_n \} \), we set a range \([a, b]\) which include a \( \xi_i \), by theorem 1, if we can find a \( x_0 \) that satisfied \( |\xi_i - x_0| \leq \min\{ (\xi_i - a), (b - \xi_i) \} \), let \( x_n = f(x_{n-1}) \) 

\( (n = 0, 1, 2, L) \) be tenable, then we can conclude \( \lim_{n \to \infty} x_n = \xi_i \). By the theorem 4, if we can find a \( x_0 \) in \([a, b]\), let \( x_n = f(x_{n-1}) \) \( (n = 0, 1, 2, L) \) be tenable, then we can conclude \( \lim_{n \to \infty} x_n = \xi_i \). If \( |f'(\xi_i)| > 1 \) \( (i = 1, 2, L) \), we can know sequence is divergent by theorem 4. If theorem 1, theorem 2, theorem 4 can not solve the question, we can use theorem 3 solve it. And when \( |f'(\xi_i)| = 1 \), the convergence of sequence \( \{ x_n \} \) is uncertain.

REFERENCES