Existence of positive solutions for boundary value problems for a class of nonlinear high-order differential equation

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Abstract: In this paper, we investigate a kind of boundary value problem of nonlinear high-order differential equation. Making use of the fixed point theorems on cone and by constructing the correction function, we obtain the existence of positive solutions for it.

Keywords: Nonlinear high-order differential equation, boundary value problem, Positive solution, Fixed point theorems on cone.
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INTRODUCTION

In recent years, the existence and multiplicity of solutions for boundary value problems for nonlinear high-order ordinary differential equations, especially for even number order equation were widely investigated, and gave a lot of satisfactory results on condition of the conjugate boundary conditions or simpler boundary conditions in papers [1-4].

In present paper, by constructing the correction function, we investigate a kind of boundary value problem of nonlinear high-order differential equation with different boundary conditions, and based on the theorem of Krasnosellskii we obtain the existence of positive solutions for it.

Preliminary Notes

In this paper, we concern on the existence of positive solutions for the following nonlinear higher-order boundary value problem

\[
\begin{aligned}
(-1)^n y^{(2n)}(x) &= f(x, y(x)), \quad 0 < x < 1 \\
y^{(i)}(0) &= 0, \quad 0 \leq i \leq k - 1 \\
y^{(j)}(1) &= 0, \quad k \leq j \leq 2n - 1
\end{aligned}
\]  

(1)

where \( n \geq 2, 1 \leq k \leq 2n - 1 \).

We assume that

\( (H_1) \ f \in C([0,1] \times [0, +\infty], (-\infty, +\infty)) \) is continuous and nonnegative.

\( (H_2) \ \exists N(x) \in L^1(0,1), N(x) > 0 \) and \( 0 < \int_0^1 g(s)N(s)ds < +\infty \) satisfying

\[ f(x, y) + N(x) \geq 0 \]

when \( y \geq 0 \) for any \( x \in (0,1) \).

Let \( I = [0,1], E = C[I, R] \), then \( E \) be a Banach space with \( \| u \| = \max_{x \in I} |u(x)| \).

In addition, we introduce space \( L^1[I,0,1] \) with norm \( \| u \| = \int_0^1 |u(x)| \, dx \).
Theorem

Let $B$ be a Banach space and $K \subset B$ a cone in $B$, and $\Omega_1, \Omega_2$ be open bounded subsets of $B$ with $0 \in \Omega_1$. Assume that $\Phi : K \cap (\Omega_2 \setminus \Omega_1) \to K$ be a completely continuous operator satisfying the condition

(i) $\|\Phi y\| \leq \|y\|, y \in K \cap \partial \Omega_1, \|\Phi y\| \geq \|y\|, y \in K \cap \partial \Omega_2$; or

(ii) $\|\Phi y\| \geq \|y\|, y \in K \cap \partial \Omega_1, \|\Phi y\| \leq \|y\|, y \in K \cap \partial \Omega_2$.

then $\Phi$ must have at least one fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$.

MAIN RESULTS

Theorem \[5\] Let $G(t,s)$ be Grenn function for boundary value problem (1), then

$G(x,s) = \begin{cases} \frac{1}{[(n-1)!]^2} \int_0^x u^{n-1} (u+1-x)^{n-1} du, & 0 \leq s \leq x \leq 1 \\ \frac{1}{[(n-1)!]^2} \int_0^x u^{n-1} (u+1-x)^{n-1} du, & 0 \leq x \leq s \leq 1 \end{cases}$

with

$\alpha(x) g(s) \leq G(x,s) \leq \beta(x) g(s) \left| \frac{\partial G(x,s)}{\partial x} \right| \leq \frac{2^{n-1}}{[(n-1)!]^2} s^{n-1} = cs^{n-1} \quad (2)$

where $\alpha(x) = \frac{x^n}{2n-1}, \beta(x) = \frac{x^{n-1}}{n}, g(s) = \frac{1}{[(n-1)!]^2} s^n$.

Lemma

For Grenn function for boundary value problem (1), we have

$G(x,s) \leq \frac{1}{[(n-1)!]^2} G(x,s) \leq \frac{2n-1}{[(n-1)!]^2} \alpha(x) \quad (3)$

for any $x, s \in [0,1]$.

Proof

We can easily obtain $G(x,s) \leq g(s) \leq \frac{1}{[(n-1)!]^2}$,

$G(x,s) = \begin{cases} \frac{1}{[(n-1)!]^2} \int_0^x u^{n-1} (u+1-x)^{n-1} du, & 0 \leq s \leq x \leq 1 \\ \frac{1}{[(n-1)!]^2} \int_0^x u^{n-1} (u+1-x)^{n-1} du, & 0 \leq x \leq s \leq 1 \end{cases}$

$\leq \frac{1}{[(n-1)!]^2} \int_0^x x^{n-1} dy = \frac{1}{[(n-1)!]^2} x^n, & 0 \leq s \leq x \leq 1 

\leq \frac{1}{[(n-1)!]^2} \int_0^x y^{n-1} dy = \frac{1}{[(n-1)!]^2} \frac{x^n}{n}, & 0 \leq x \leq s \leq 1

\leq \frac{x^n}{[(n-1)!]^2} = \frac{2n-1}{[(n-1)!]^2} \alpha(x)$

Lemma

If $y(x) \in C^n[0,1]$ satisfying the conditions as follows
\[
\begin{cases}
(-1)^n y^{(2n)} (x) = h(x), \quad 0 < x < 1 \\
y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1 \\
y^{(j)}(1) = 0, \quad k \leq j \leq 2n - 1,
\end{cases}
\]
then we have \( y(x) \leq \| y(x) \| \alpha(x), 0 \leq x \leq 1 \), where \( h(x) \geq 0 \).

**Proof**

By theorem 2.2 and lemma 2.1, we get
\[
\| y(x) \| = \max_{0 \leq x \leq 1} \int_0^1 G(x,s)h(s)ds \leq \int_0^1 g(s)h(s)ds.
\]
and so we can obtain
\[
y(x) = \int_0^1 G(x,s)h(s)ds \geq \alpha(x) \int_0^1 g(s)h(s)ds \geq \| y \| \alpha(x).
\]

**Lemma**

Assume that \((H_1)\) and \((H_2)\) hold with \( \omega(x) \in C^n[0,1] \) satisfying
\[
\begin{cases}
(-1)^n y^{(2n)} (x) = N(x), \quad 0 < x < 1 \\
y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1 \\
y^{(j)}(1) = 0, \quad k \leq j \leq 2n - 1
\end{cases}
\]
that there exists a constant \( C \) such that \( \omega(x) \leq C\alpha(x) \), where \( N(x) > 0, 0 \leq x \leq 1 \).

**Proof**

For any \( x \in [0,1] \), from Lemma 2.1, we have
\[
\omega(x) = \int_0^1 G(x,s)N(s)ds \leq \frac{(2n-1)}{(n-1)!} \| N \| \alpha(x) = C\alpha(x),
\]
where \( C = \frac{2n-1}{[(n-1)!]^2} \| N \| \).

For any \( x \in I \), we define the operator correction function as
\[
F(x,y) = g(x,y) + N(x), \quad g(x,y) = \begin{cases} f(x,y), y \geq 0, \\
f(x,0), y < 0. \end{cases}
\]

In this paper, we concern on the modified boundary value problem
\[
\begin{cases}
(-1)^n y^{(2n)} (x) = F(x, y(x) - \omega(x)), \quad 0 < x < 1 \\
y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1 \\
y^{(j)}(1) = 0, \quad k \leq j \leq 2n - 1
\end{cases}
\]

\( \omega(x) \) defined by Lemma 2.3.

**Lemma**

If \( u(x) = y(x) + \omega(x) \) is a solution for problem (4) with \( y(x) \geq 0 \) for any \( x \in [0,1] \), then \( y(x) \) must be the positive solution for problem (1).

**Proof**

For any \( x \in [0,1] \), if \( u(x) = y(x) + \omega(x) \) is a solution for problem (4), then by definition of \( F(x, y) \), we have
\[
\begin{cases}
(-1)^n [y^{(2n)}(x) + \omega^{(2n)}(x)] = F(x, y(x)), \quad 0 < x < 1 \\
(y + \omega)^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1 \\
(y + \omega)^{(j)}(1) = 0, \quad k \leq j \leq 2n - 1
\end{cases}
\]
that is

Available Online: [http://saspjournals.com/sjpms](http://saspjournals.com/sjpms)
\[\begin{align*}
&(-1)^n y^{(2n)}(x) = f(x, y(x)), 0 < x < 1 \\
y^{(i)}(0) = 0, 0 \leq i \leq k - 1 \\
y^{(j)}(1) = 0, k \leq j \leq 2n - 1
\end{align*}\]

So \(y(x)\) is the positive solution for problem (1).

Clearly, the problem (4) is equivalent to the integral equation

\[y(x) = \int_0^1 G(x, s)F(s, y(s) - \alpha(s))ds\]

We define the operator \(\Phi: K \to K\) by

\[(\Phi y)(x) = \int_0^1 G(x, s)F(s, y(s) - \alpha(s))ds\]

We define the cone \(K = \{ y \in E : y(x) \geq \|\alpha(x)\|, x \in [\theta, 1 - \theta]\}, \)

\(\text{Let } \sigma = \frac{\theta^n}{2(2n - 1)}\).

**Lemma**

Assume that \((H_1)\) and \((H_2)\) hold, then \(\Phi(K) \subset K\), and \(\Phi: K \to K\) is completely continuous.

**Proof**

For any \(y \in K, x \in [0, 1]\), from (2) and (3) we have

\[\|\Phi y\| \leq \int_0^1 g(s)F(s, y(s) - \alpha(s))ds\]

thus

\[\Phi y(x) \geq \alpha(x)\int_0^1 g(s)F(s, y(s) - \alpha(s))ds \geq \|\Phi y\| \alpha(x)\]

This leads to \(\Phi y(x) \geq \alpha(x)\|\Phi y\|\). Thus we get \(y \in K\), that is, \(\Phi(K) \subset K\).

By the Ascoli-Arzelà theorem, we can easily see that \(\Phi: K \to K\) is completely continuous.

**Theorem**

Assume that \((H_1)\) and \((H_2)\) hold, with the function \(f\) satisfying

\((\text{I})\) \(\forall (x, y) \in [0, 1] \times [\sigma R, R], \text{ such that } f(x, y) \geq MR;\)

\((\text{II})\) \(\forall (x, y) \in [0, 1] \times [0, r], \text{ such that } f(x, y) \leq mr,\)

for any \(C < r < 2C < R, \text{ where } m \leq (\int_0^1 (g(s) + N(s))ds)^{-1}, M \geq (2\sigma \int_0^{\theta - \theta} g(s)ds)^{-1}\) and \(mr \geq 1,\) then problem (1) has at least one positive solution \(y \in K\) satisfying \(r \leq y \leq R\).

**Proof**

Let \(\Omega_1 = \{ y \in K : \|y\| < R\}, \text{ then for any } x \in \partial\Omega_1, x \in [0, 1]\) we have

\[y(x) - \alpha(x) \geq y(x) - CA(x) \geq y(x)(1 - \frac{C}{R}) \geq y(x)(1 - \frac{C}{2R}) \geq \frac{1}{2} \|\alpha(x)\| = \frac{1}{2} \alpha(x)R.\]

And for any \(x \in \partial\Omega_1, x \in [\theta, 1 - \theta]\), we have

\[\sigma R = \frac{\theta^n}{2(2n - 1)} R \leq \frac{R}{2} \alpha(x) \leq x(x) - \alpha(x) \leq \|x\| = R.\]

Thus, we obtain

\[\|\Phi y(x)\| \geq \int_0^{\theta - \theta} G(x, s)F(s, y(s) - \alpha(s))ds \geq \int_0^{\theta - \theta} G(x, s)[p(s)f(s, y(s) - \alpha(s))]ds \geq \alpha(x)MR\int_0^{\theta - \theta} g(s)ds \geq 2\sigma(2\sigma \int_0^{\theta - \theta} g(s)ds)^{-1} R \cdot \int_0^{\theta - \theta} g(s)ds = R = \|y\|\]

Let \(\Omega_2 = \{ y \in K : \|y\| < r\}, \text{ then for any } y \in \partial\Omega_2, x \in [0, 1]\), we can see that
y(x) - \omega(x) \leq y(x) \leq r

and

y(x) - \omega(x) \geq y(x) - C\alpha(x) \geq y(x)(1 - C/r) \geq 0

Therefore \|\Phi y(x)\| \leq \int_0^1 g(s)F(s,u(s) - \omega(s))ds

\leq \int_0^1 [g(s)mr + N(s)]ds = \|y\| \leq mr\int_0^1 [g(s) + N(s)]ds \leq r = \|y(x)\|

Thus, problem (1) has at least one positive solution y \in K satisfying r \leq y \leq R.

From Lemma 2.3, we can see for any \theta \leq x \leq 1 - \theta that

u(x) \geq \|u\| \alpha(x) \geq r\alpha(x) > C\alpha(x) \geq \omega(x)

that is, y(x) = u(x) - \omega(x) is the positive solution for problem (1).

Theorem

Assume that (H_1) and (H_2) hold with

(\text{III}) 2(\sigma^2\int_0^{1-\theta} g(s)ds)^{-1} < \lim_{y \to +\infty} \min_{x \in (0,1]} \frac{f(x,y)}{y} < +\infty;

(\text{IV}) 0 \leq \lim_{y \to 0} \max_{x \in (0,1]} \frac{f(x,y)}{y} < (\int_0^1 [g(s) + \omega(s)]ds)^{-1},

then problem (1) has at least one positive solution y \in K satisfying r \leq y \leq R.

Proof

Let A = \lim_{y \to +\infty} \min_{x \in (0,1]} \frac{f(x,y)}{y}. Owing to condition(\text{III}) we can see by taking

\varepsilon = A - 2(\sigma^2\int_0^{1-\theta} g(s)ds)^{-1} > 0,

that there exists a sufficiently large number R \neq r, such that

f(x,y) \geq (A - \varepsilon)y \geq (\sigma^2\int_0^{1-\theta} g(s)ds)^{-1}\sigma R = (\sigma\int_0^{1-\theta} g(s)ds)^{-1}R

for any y \geq \sigma R.

Let M = (\sigma\int_0^{1-\theta} g(s)ds)^{-1}, then we have

M > (2\sigma\int_0^{1-\theta} g(s)ds)^{-1} and f(x,y) \geq MR, \sigma R \leq y \leq R.

Let B = \lim_{y \to 0} \max_{x \in (0,1]} \frac{f(x,y)}{y}. Owing to condition (\text{IV}), we can see by taking \varepsilon = (\int_0^1 [g(s) + \omega(s)]ds)^{-1} - B > 0

, For any 0 \leq y \leq \rho, if y \neq 0, we get

f(x,y) \leq (B + \varepsilon)y \leq (\int_0^1 [g(s) + \omega(s)]ds)^{-1}\rho

By letting m = (\int_0^1 [g(s) + \omega(s)]ds)^{-1}, r = \rho, we get f(x,y) \leq mr, 0 \leq y \leq r.

Thus, we see from Theorem 2.3 that problem (1) has at least one positive solution y \in K satisfying r \leq y \leq R.

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