On The Binary Quadratic Diophantine Equation $x^2 - 4xy + y^2 + 14x = 0$

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Abstract: The binary quadratic equation $x^2 - 4xy + y^2 + 14x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Keywords: Binary quadratic equation, integral solutions.

INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety [1, 2, 3, 4, 5, 6]. In the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by $x^2 - 4xy + y^2 + 14x = 0$. The recurrence relations satisfied by the solutions $x$ and $y$ are given. Also a few interesting properties among the solutions are exhibited [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 4xy + y^2 + 14x = 0 \quad (1)$$

Note that (1) is satisfied by the following non-zero integer pairs

$$(1,7), (4,-56), (3,7), (4,-14)$$

However, we have other solutions for (1), which are illustrated below:

Solving (1) for $y$, we have

$$y = 2x \pm \sqrt{3x^2 - 14x} \quad (2)$$

Let $\alpha = 3\beta - 14x$

Multiplying the above equation by 3 on both sides and performing a few calculations, we have

$$X^2 = 3\alpha^2 + 49 \quad (3)$$

where

$$X = 3x - 7 \quad (4)$$

The least positive integer solution of (3) is $\alpha_0 = 7, X_0 = 14$

Now, to find the other solution of (3), consider the pellian equation

$$X^2 = 3\alpha^2 + 1 \quad (5)$$

whose fundamental solution is

$$\left(\tilde{a}_0, \tilde{X}_0\right) = (1,2)$$

The other solutions of (5) can be derived from the relations

$$\tilde{X}_n = \frac{f_n + \tilde{a}_n}{2}, \quad \tilde{a}_n = \frac{g_n}{2\sqrt{3}}$$

where

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\[ f_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \]
\[ g_n = (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \]

Applying the lemma of Brahmagupta between \( (\alpha, X_n) \) & \( (\tilde{\alpha}, \tilde{X}_n) \), the other solutions of (3) can be obtained from the relation

\[ \alpha_{n+1} = \frac{7}{2} f_n + \frac{7}{\sqrt{3}} g_n \]  \hspace{1cm} (6)
\[ X_{n+1} = 7 f_n + \frac{21}{2\sqrt{3}} g_n \]  \hspace{1cm} (7)

Taking positive sign on the R.H.S of (2) and using (4),(6)&(7), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows

\[ x_{n+1} = \frac{1}{3} \left( 7 f_n + \frac{21}{2\sqrt{3}} g_n + 7 \right) \]  \hspace{1cm} (8)
\[ y_{n+1} = 2 x_{n+1} + \left( \frac{7}{2} f_n + \frac{7}{\sqrt{3}} g_n \right) n = -1, 1, 3, 5, ... \]  \hspace{1cm} (9)

The recurrence relations for \( x_{n+1}, y_{n+1} \) are respectively

\[ 6 x_{n+1} - 84 x_{n+5} + 6 x_{n+5} = -168 \]
\[ 6 y_{n+1} - 84 y_{n+3} + 6 y_{n+5} = -336 \]

A few numerical examples are given in table below

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_{n+1} )</th>
<th>( y_{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>7</td>
<td>21</td>
</tr>
<tr>
<td>1</td>
<td>63</td>
<td>231</td>
</tr>
<tr>
<td>3</td>
<td>847</td>
<td>3157</td>
</tr>
<tr>
<td>5</td>
<td>11767</td>
<td>43911</td>
</tr>
</tbody>
</table>

Some relations satisfied by the solutions (8) & (9) are as follows

1. \( x_{n+3} = -x_{n+1} + 4 y_{n+1} - 14 \)
2. \( x_{n+5} = 56 y_{n+1} - 15 x_{n+1} - 224 \)
3. \( y_{n+3} = 15 y_{n+1} - 4 x_{n+1} - 56 \)
4. \( x_{n+5} = 209 y_{n+1} - 56 x_{n+1} - 840 \)
5. \( x_{n+3} = 15 x_{n+3} - 4 y_{n+3} - 14 \)
6. \( x_{n+5} = 4 y_{n+3} - x_{n+3} - 14 \)
7. \( y_{n+3} = 4 x_{n+3} - y_{n+3} \)
8. \( y_{n+5} = -4 x_{n+3} + 15 y_{n+3} - 56 \)
9. \( x_{n+1} = 209 x_{n+5} - 56 y_{n+5} - 224 \)
10. \( x_{n+3} = 15 x_{n+5} - 4 y_{n+5} - 14 \)
11. \( y_{n+1} = 56 x_{n+5} - 15 y_{n+5} - 56 \)
12. \( y_{n+3} = 4 x_{n+5} - y_{n+5} \)
13. Each of the following expressions is a nasty number
   i) \( 24 x_{2n+2} - 6 y_{2n+2} - 14 \)
   ii) \( 2352 x_{2n+4} - 630 y_{2n+4} - 2450 \)
REMARKABLE OBSERVATIONS
1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for hyperbolas

Example 1) Define \( x = 24x_{n+1} - 6y_{n+1} - 28, \ Y = 12y_{n+1} - 42x_{n+1} + 42 \)
Note that the pair \((X, Y)\) satisfies the hyperbola \( Y^2 = 3X^2 - 12 \times 7^2 \)

Example 2) Define \( x = 2352x_{n+3} - 630y_{n+3} - 2548, \ Y = 1092y_{n+3} - 4074x_{n+3} + 4410 \)
Note that the pair \((X, Y)\) satisfies the hyperbola \( Y^2 = 3X^2 - 12 \times 49^2 \)

2) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for parabolas

Example 3) Define \( x = 24x_{n+1} - 6y_{n+1} - 28, \ Y = 12y_{n+1} - 42x_{n+1} + 42 \)
Note that the pair \((X, Y)\) satisfies the parabola \( Y^2 = 3X^2 - 12 \times 7^2 \)

Example 4) Define \( x = 2352x_{n+3} - 630y_{n+3} - 2548, \ Y = 1092y_{n+3} - 4074x_{n+3} + 4410 \)
Note that the pair \((X, Y)\) satisfies the parabola \( Y^2 = 49 \times 3X - 12 \times 49^2 \)

Solving (1) for \( x \), we have
\[
x = 2y - 7 \pm \sqrt{3y^2 - 28y + 49}
\]

10) Let \( \alpha^2 = 3y^2 - 28y + 49 \)
Multiplying the above equation by 3 on both sides and performing a few calculations, we have
\[
Y^2 = 3\alpha^2 + 49
\]
where \( Y = 3y - 14 \)

11) The least positive integer solution of (3) is \( \alpha_0 = 7, Y_0 = 14 \)

12) Now, to find the other solution of (11), consider the pellian equation
\[
Y^2 = 3\alpha^2 + 1
\]
whose fundamental solution is \( \left(\alpha_0, Y_0\right) = (1, 2) \)

13) The other solutions of (13) can be derived from the relations
\[
\tilde{Y}_n = \frac{f_n}{2}, \tilde{\alpha}_n = \frac{g_n}{2\sqrt{3}}
\]
where
\[
f_n = (2 + \sqrt{3})^{n+1} + (2 - \sqrt{3})^{n+1}
\]
\[
g_n = (2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}
\]
Applying the lemma of Brahmagupta between \( \left(\alpha_0, Y_0\right) \) & \( \left(\tilde{\alpha}_n, \tilde{Y}_n\right) \), the other solutions of (11) can be obtained from the relation
\[
\alpha_{n+1} = \frac{7}{2} f_n + \frac{7}{2\sqrt{3}} g_n
\]
\[
Y_{n+1} = 7 f_n + \frac{21}{2\sqrt{3}} g_n
\]

14) Taking positive sign on the R.H.S of (10) and using (12),(14)&(15), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows
\[
y_{n+1} = \frac{1}{3} \left(7 f_n + \frac{21}{2\sqrt{3}} g_n + 14\right)
\]

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\[ x_{n+1} = 2y_{n+1} - 7 + \left( \frac{7}{2} f_n + \frac{7}{\sqrt{3}} g_n \right) , \]
\[ n = 0, 2, 4, 6 \ldots \]

The recurrence relations for \( x_n, y_n \) are respectively
\[ 6x_{n+1} - 84x_{n+3} + 6x_{n+5} = -168 \]
\[ 6y_{n+1} - 84y_{n+3} + 6y_{n+5} = -336 \]

A few numerical examples are given in the table below:

<table>
<thead>
<tr>
<th>Table 2: Numerical Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

Some relations satisfied by the solutions (16) & (17) are as follows
1. \[ y_{n+1} = 4x_{n+1} - y_{n+1} \]
2. \[ y_{n+5} = 56x_{n+1} - 15y_{n+1} - 56 \]
3. \[ x_{n+1} = 15x_{n+1} - 4y_{n+1} - 14 \]
4. \[ x_{n+5} = 209x_{n+1} - 56y_{n+1} - 224 \]
5. \[ y_{n+1} = 15y_{n+1} - 4x_{n+1} - 56 \]
6. \[ y_{n+5} = 4x_{n+1} - y_{n+1} \]
7. \[ x_{n+1} = 4y_{n+1} - x_{n+1} - 14 \]
8. \[ x_{n+5} = -4y_{n+1} + 15x_{n+1} - 14 \]
9. \[ y_{n+1} = 209y_{n+1} - 56x_{n+1} - 840 \]
10. \[ y_{n+3} = 15y_{n+5} - 4x_{n+5} - 56 \]
11. \[ x_{n+1} = 56y_{n+1} - 15x_{n+1} - 224 \]
12. \[ x_{n+3} = 4y_{n+5} - x_{n+5} - 14 \]
13. Each of the following expressions is a nasty number
   i) \[ 24y_{2n+2} - 6x_{2n+2} - 84 \]
   ii) \[ 2352y_{n+3} - 630x_{n+3} - 9408 \]

REMARKABLE OBSERVATIONS

By considering suitable linear transformations between the solutions of (1), one may get integer solutions for hyperbolas

Example 5) Define \( X = 24y_{n+1} - 6x_{n+1} - 98 \). \( Y = 12x_{n+1} - 42y_{n+1} + 168 \)

Note that the pair (X, Y) satisfies the hyperbola \[ Y^2 = 3X^2 - 12 \times 7^2 \]

Example 6) Define \( X = 2352y_{n+3} - 630x_{n+3} - 9506 \). \( Y = 1092x_{n+3} - 4074y_{n+3} + 16464 \)

Note that the pair (X, Y) satisfies the hyperbola \[ Y^2 = 3X^2 - 12 \times 49^2 \]

By considering suitable linear transformations between the solutions of (1), one may get integer solutions for parabolas

Example 7) Define \( X = 24y_{n+1} - 6x_{n+1} - 98 \). \( Y = 12x_{n+1} - 42y_{n+1} + 168 \)

Note that the pair (X, Y) satisfies the parabola \[ Y^2 = 7 \times 3X - 12 \times 7^2 \]
Example 8) Define \( X = 2352y_{n+3} - 630x_{n+3} - 9506 \), \( Y = 1092x_{n+3} - 4074y_{n+3} + 16464 \).

Note that the pair \( (X, Y) \) satisfies the parabola \( Y^2 = 49 \times 3X - 12 \times 49^2 \).

CONCLUSION

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

ACKNOWLEDGEMENT

The financial support from the UGC, New Delhi (F.MRP-5123/14 (SERO/UGC) dated March 2014) for a part of this work is gratefully acknowledged.

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