A Specific Formula to Compute the Determinant of One Matrix of Order \( n \)

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Abstract: Let \( A = [\alpha_{ij}] \) be an \( n \times n \) matrix, where \( \alpha_{ij} = \frac{1}{a_i + b_j} \), \( i, j = 1, 2, \ldots, n \). In this paper, we establish a specific formula to calculate the determinant of matrix \( A \).

Keywords: Determinant; Matrix; Laplace Theorem.

Introduction

Determinants occur throughout mathematics. For example, a matrix is often used to represent the coefficients in a system of linear equations, and the determinant can be used to solve those equations, although more efficient techniques are actually used, some of which are determinant-revealing and consist of computationally effective ways of computing the determinant itself. For an \( n \times n \) matrix \( A \), its determinant is defined as

\[
|A| = \sum_{\sigma} \text{sign}(\sigma) \prod_{i=1}^{n} \alpha_{\sigma(i)},
\]

where the sum runs over all \( n! \) permutations \( \sigma \) of the \( n \) items \( 1, 2, \ldots, n \) and the \text{sign} of a permutation \( \sigma \), \( \text{sign}(\sigma) \) is \( +1 \) or \( -1 \), according to whether the minimum number of transpositions, or pair-wise interchanges, necessary to achieve it starting from \( 1, 2, \ldots, n \) is even or odd. Thus, each product

\[
\prod_{i=1}^{n} \alpha_{\sigma(i)}
\]

enters into the determinant with a + sign if the permutation \( \sigma \) is even or a − sign if it is odd. The most fundamental and naive method of implementing an algorithm to compute the determinant is to use Laplace's formula [1] for expansion by cofactors, i.e.,

\[
|A| = \sum_{j=1}^{n} \alpha_{ij} A_{ij}, i = 1, 2, \ldots, n,
\]

where \( A_{ij} \) which is called the cofactor of \( \alpha_{ij} \), is a product of \( (-1)^{i+j} \) and the minor resulting from the deletion of row \( i \) and column \( j \). This approach is extremely inefficient in general, however, as it is of order \( n! \) for an \( n \times n \) matrix. Consequently, those determinants which have special constructors are investigated. There are a series of literatures about this topic, such as the referenced [2–6] and the references therein.

In this paper, we focus on an \( n \times n \) matrix \( A = [\alpha_{ij}]_{n \times n} \), where \( \alpha_{ij} = \frac{1}{a_i + b_j} \), \( i, j = 1, 2, \ldots, n \). One specific formula to calculate the determinant of matrix \( A \) is established.

Main result and its proof

To state clearly, let \( D_n \) be the determinant of the \( n \times n \) matrix \( A = [\alpha_{ij}] \). For \( n = 1 \), the conclusion is trivial. In general, assume that \( n \geq 2 \). Our main result is to establish a specific formula to compute \( D_n \).
Theorem 1. For \( n \geq 2 \),
\[
D_n = \frac{\prod_{j=2}^{n} (a_j-a_i)(b_j-b_i)}{\prod_{j=1}^{n} (a_j+b_j)}.
\]

Proof. We complete the proof by induction on the order \( n \) of matrix \( A \). For \( n = 2 \), we obtain
\[
D_2 = \begin{pmatrix}
1 & 1 \\
\frac{1}{a_1+b_1} & \frac{1}{a_1+b_2}
\end{pmatrix}
\]
\[
= \frac{1}{a_1+b_1} \times \frac{1}{a_2+b_1} - \frac{1}{a_1+b_2} \times \frac{1}{a_2+b_1}
\]
\[
= (a_1+b_2)(a_2+b_1) - (a_1+b_1)(a_2+b_2)
\]
\[
= (a_1+b_2)(a_2+b_1)(a_1+b_2)(a_2+b_1)
\]
\[
= (a_1-a_2)(b_1-b_2).
\]

It follows that Theorem 1 holds when \( n = 2 \).

Now, we assume that Theorem 1 holds when \( n = k \), where \( k \geq 2 \). That is to say,
\[
D_k = \frac{\prod_{j=2}^{k} (a_j-a_i)(b_j-b_i)}{\prod_{j=1}^{k} (a_j+b_j)}.
\]

Then when \( n = k+1 \),
\[
D_{k+1} = \begin{pmatrix}
1 & 1 & 1 \\
\frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \frac{1}{a_1+b_{k+1}} \\
\frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \frac{1}{a_2+b_k} \\
\vdots & \ddots & \vdots \\
\frac{1}{a_{k+1}+b_1} & \frac{1}{a_{k+1}+b_2} & \frac{1}{a_{k+1}+b_{k+1}}
\end{pmatrix}
\]

By adding column 1 multiplied by a scalar \(-1\) to column \( j \) for \( j = 2, 3, \ldots, k+1 \), we obtain that
By adding row 1 multiplied by a scalar −1 to row \( j \), \( j = 2, 3, \ldots, k+1 \), we obtain that

\[
D_{k+1} = \begin{vmatrix}
1 & \frac{b_1 - b_2}{a_1 + b_1} & \cdots & \frac{b_1 - b_{k+1}}{a_1 + b_1} \\
a_2 + b_1 & \frac{b_1 - b_2}{a_2 + b_2} & \cdots & \frac{b_1 - b_{k+1}}{a_2 + b_2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k+1} + b_1 & \frac{b_1 - b_2}{a_{k+1} + b_{k+1}} & \cdots & \frac{b_1 - b_{k+1}}{a_{k+1} + b_{k+1}}
\end{vmatrix}
\]

\[
= \frac{1}{\prod_{i=1}^{k+1} (a_i + b_i)} \begin{vmatrix}
1 & \frac{1}{a_1 + b_1} & \cdots & \frac{1}{a_1 + b_{k+1}} \\
\frac{a_2 - a_1}{a_2 + b_2} & \frac{1}{a_2 + b_2} & \cdots & \frac{1}{a_2 + b_{k+1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_{k+1} - a_k}{a_{k+1} + b_{k+1}} & \frac{1}{a_{k+1} + b_{k+1}} & \cdots & \frac{1}{a_{k+1} + b_{k+1}}
\end{vmatrix}
\]

By Laplacian Theorem and row-multiplying transformations, we have

\[
D_{k+1} = \frac{\prod_{i=2}^{k+1} (a_i - a_{i-1})(a_i + a_{i-1})}{(a_1 + b_1)\prod_{i=2}^{k+1} (a_i + b_i)(a_i + b_{i-1})} \begin{vmatrix}
1 & \frac{1}{a_2 + b_2} & \cdots & \frac{1}{a_2 + b_{k+1}} \\
\frac{a_3 - a_2}{a_3 + b_3} & \frac{1}{a_3 + b_3} & \cdots & \frac{1}{a_3 + b_{k+1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_{k+1} - a_{k}}{a_{k+1} + b_{k+1}} & \frac{1}{a_{k+1} + b_{k+1}} & \cdots & \frac{1}{a_{k+1} + b_{k+1}}
\end{vmatrix}
\]

It is clear that

\[
\begin{vmatrix}
1 & \frac{1}{a_2 + b_2} & \cdots & \frac{1}{a_2 + b_{k+1}} \\
\frac{a_3 - a_2}{a_3 + b_3} & \frac{1}{a_3 + b_3} & \cdots & \frac{1}{a_3 + b_{k+1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_{k+1} - a_{k}}{a_{k+1} + b_{k+1}} & \frac{1}{a_{k+1} + b_{k+1}} & \cdots & \frac{1}{a_{k+1} + b_{k+1}}
\end{vmatrix}
\]

is a determinant of order \( k \). By the assumption, we have
\[
\begin{array}{cccc}
1 & & 1 \\
\frac{a_2 + b_2}{a_{k+1} + b_2} & \ddots & \frac{a_{k+1} + b_{k+1}}{a_{k+1} + b_{k+1}} \\
\vdots & \ddots & \vdots \\
\frac{a_{k+1} + b_2}{a_{k+1} + b_{k+1}} & \frac{a_2 + b_2}{a_{k+1} + b_{k+1}} & \frac{1}{1}
\end{array}
\]

Consequently,

\[
D_{k+1} = \frac{\prod_{i=2}^{k+1} (b_i - a_i)(a_i - a_i) \prod_{j=2}^{k+1} (a_j - a_j)(b_j - b_j)}{(a_i + b_i) \prod_{i=2}^{k+1} (a_i + b_i)(a_i + b_i) \prod_{j=2}^{k+1} \prod_{i=2}^{k+1} (a_i + b_j)}.
\]

Simplifying the above equality leads to

\[
D_{k+1} = \frac{\prod_{j=2}^{k+1} (a_j - a_i)(b_j - b_i)}{\prod_{j=1}^{k+1} \prod_{i=1}^{k+1} (a_i + b_j)}.
\]

By induction, we obtain that for \( n \geq 2 \),

\[
D_n = \frac{\prod_{j=2}^{n} (a_j - a_i)(b_j - b_i)}{\prod_{j=1}^{n} \prod_{i=1}^{n} (a_i + b_j)}.
\]

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