More on gbsb*-Closed Sets in Topological Spaces

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Abstract: Using the concept of gbsb*-open sets and gbsb*-closed sets, we introduce and study the topological properties of gbsb*-interior and gbsb*-closure of a set, gbsb*-derived sets, gbsb*-border, gbsb*-frontier and gbsb*-exterior.

Keywords: gbsb*-interior, gbsb*-closure, gbsb*-limit point, gbsb*-derived set, gbsb*-border, gbsb*-frontier, gbsb*-exterior

INTRODUCTION

The notion of generalized b-strongly b*-closed set and its different characterizations are given in [4]. In this paper, we introduce the notions of gbsb*-limit points, gbsb*-derived sets, gbsb*-interior and gbsb*-closure of a set, gbsb*-interior points, gbsb*-border, gbsb*-frontier and gbsb*-exterior by using the concept of gbsb*-open sets and gbsb*-closed sets, and study their topological properties.

PRELIMINARIES

Throughout this paper \((X, \tau)\) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. \((X, \tau)\) will be replaced by \(X\) if there is no changes of confusion. For a subset \(A\) of a topological space \(X\), \(\text{cl}(A)\) and \(\text{int}(A)\) denote the closure of \(A\) and the interior of \(A\) respectively. We recall the following definitions and results.

Definition 2.1.[1] Let \((X, \tau)\) be a topological space. A subset \(A\) of the space \(X\) is said to be \(b\)-open if \(A \subseteq \text{int}(\text{cl}(A))\cup \text{cl}(\text{int}(A))\) and \(b\)-closed if \(\text{int}(\text{cl}(A))\cap \text{cl}(\text{int}(A)) \subseteq A\).

Definition 2.2.[1] Let \((X, \tau)\) be a topological space and \(A \subseteq X\). The \(b\)-closure of \(A\), denoted by \(b\text{cl}(A)\) and is defined by the intersection of all \(b\)-closed sets containing \(A\).

Definition 2.3.[1] Let \((X, \tau)\) be a topological space and \(A \subseteq X\). The \(b\)-interior of \(A\), denoted by \(b\text{int}(A)\) and is defined by the union of all \(b\)-open sets contained in \(A\).

Definition 2.4.[2] A subset \(A\) of a topological space \((X, \tau)\) is said to be generalized \(b\)-closed (briefly \(gb\)-closed) if \(b\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(b\)-open in \((X, \tau)\). The collection of all \(gb\)-closed sets of \(X\) is denoted by \(gb\text{-C}(X, \tau)\).

Definition 2.5.[2] Let \((X, \tau)\) be a topological space and \(A \subseteq X\). The \(gb\)-closure of \(A\), denoted by \(gb\text{-cl}(A)\) and is defined by the intersection of all \(gb\)-closed sets containing \(A\).

Definition 2.6.[2] Let \((X, \tau)\) be a topological space and \(A \subseteq X\). The \(gb\)-interior of \(A\), denoted by \(gb\text{-int}(A)\) and is defined by the union of all \(gb\)-open sets contained in \(A\).

Definition 2.7.[3] Let \((X, \tau)\) be a topological space. A subset \(A\) of \(X\) is said to be strongly \(b\*-closed\) (briefly \(sb\*-closed\)) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(b\)-open in \((X, \tau)\).
Definition 2.8.[4] A subset A of a topological space \((X, \tau)\) is called a generalized b-strongly b*-closed set (briefly, gbbsb*-closed) if \(\text{bcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is gbbsb*-open in \((X, \tau)\). The collection of all gbbsb*-closed sets of \(X\) is denoted by gbbsb*-C(X,\(\tau\)).

Theorem 2.9. [4]
(i) Every closed set is gbbsb*-closed.
(ii) Every b-closed set is gbbsb*-closed.
(iii) Every gbbsb*-closed set is gb-closed.

Theorem 2.10.[4]
(i) Arbitrary intersection of gbbsb*-closed sets is gbbsb*-closed.
(ii) Arbitrary union of gbbsb*-open sets is gbbsb*-open.

Remark 2.11.[4]
(i) The union of gbbsb*-closed sets need not be a gbbsb*-closed set.
(ii) The intersection of gbbsb*-open sets need not be a gbbsb*-open set.

The generalized b-strongly b*-interior operator

Definition 3.1. Let \(A\) be a subset of a topological space \((X, \tau)\). Then the union of all gbbsb*-open sets contained in \(A\) is called the gbbsb*-interior of \(A\) and it is denoted by gbbsb*int(A). That is, gbbsb*int(A) = \(\{V : V \subseteq A\text{ and } V \in \text{gbbsb}*-O(X)\}\).

Remark 3.2. Since the union of gbbsb*-open subsets of \(X\) is gbbsb*-open in \(X\), gbbsb*int(A) is gbbsb*-open in \(X\).

Definition 3.3. Let \(A\) be a subset of a topological space \(X\). A point \(x \in X\) is called a gbbsb*-interior point of \(A\) if there exists a gbbsb*-open set \(G\) such that \(x \in G \subseteq A\).

Theorem 3.4. Let \(A\) be a subset of a topological space \((X, \tau)\). Then
(i) gbbsb*int(A) is the largest gbbsb*-open set contained in A.
(ii) A is gbbsb*-open if and only if gbbsb*int(A)=A.
(iii) gbbsb*int(A) is the set of all gbbsb*-interior points of A.
(iv) A is gbbsb*-open if and only if every point of A is a gbbsb*-interior point of A.

Proof:
(i) Being the union of all gbbsb*-open sets, gbbsb*int(A) is gbbsb*-open and contains every gbbsb*-open subset of A. Hence gbbsb*int(A) is the largest gbbsb*-open set contained in A.
(ii) Necessity: Suppose A is gbbsb*-open. Then by Definition 3.1, \(A \subseteq \text{gbbsb*int}(A)\). But gbbsb*int(A) \(\subseteq A\) and therefore gbbsb*int(A)=A. Sufficiency: suppose gbbsb*int(A)=A. Then by Remark 3.2, gbbsb*int(A) is gbbsb*-open set. Hence A is gbbsb*-open.
(iii) \(x \in \text{gbbsb*int}(A)\) if and only if there exists a gbbsb*-open set \(G\) such that \(x \in G \subseteq A\).
\[\iff x \text{ is a gbbsb*-interior point of } A.\]
Hence gbbsb*int(A) is the set of all gbbsb*-interior points of A.
(iv) It directly follows from (ii) and (iii).

Theorem 3.5. Let \(A\) and \(B\) be subsets of \((X, \tau)\). Then the following results hold.
(i) gbbsb*int\(\phi\)=\(\phi\) and gbbsb*int(X)=X.
(ii) If \(B\) is any gbbsb*-open set contained in \(A\), then \(B \subseteq \text{gbbsb*int}(A)\).
(iii) If \(A \subseteq B\), then gbbsb*int(A) \(\subseteq \text{gbbsb*int}(B)\).
(iv) int(A) \(\subseteq \text{bint}(A) \subseteq \text{gbbsb*int}(A) \subseteq \text{gbint}(A) \subseteq A\).
(v) gbbsb*int(gbbsb*int(A))=gbbsb*int(A).

Proof:
(i) Since \(\phi\) is the only gbbsb*-open set contained in \(\phi\), then gbbsb*int(\(\phi\))=\(\phi\). Since X is gbbsb*-open and gbbsb*int(X) is the union of all gbbsb*-open sets contained in X, gbbsb*int(X)=X.
(ii) Suppose B is gbbsb*-open set contained in A. Since gbbsb*int(A) is the union of all gbbsb*-open set contained in A, then we have \(B \subseteq \text{gbbsb*int}(A)\).

(iii) Suppose \( A \subseteq B \). Let \( x \in \text{gbsb*int}(A) \). Then \( x \) is a gbsb*-interior point of \( A \) and hence there exists a gbsb*-open set \( G \) such that \( x \in G \subseteq A \). Since \( A \subseteq B \), then \( x \in G \subseteq B \). Therefore \( x \) is a gbsb*-interior point of \( B \). Hence \( x \in \text{gbsb*int}(B) \). This proves (iii).

(iv) Since every gbsb*-open set is gb-open, \( \text{gbsb*int}(A) \subseteq \text{gbint}(A) \). Since every b-open set is gbsb*-open, \( \text{bint}(A) \subseteq \text{gbsb*int}(A) \). Every open set is b-open, \( \text{int}(A) \subseteq \text{bint}(A) \). Therefore \( \text{int}(A) \subseteq \text{bint}(A) \subseteq \text{gbsb*int}(A) \subseteq \text{gbint}(A) \subseteq A \).

(v) By Remark 3.2, \( \text{gbsb*int}(A) \) is gbsb*-open and by Theorem 3.4(ii), \( \text{gbsb*int}(\text{gbsb*int}(A)) = \text{gbsb*int}(A) \).

Theorem 3.6. Let \( A \) and \( B \) be the subsets of a topological space \( X \). Then,
(i) \( \text{gbsb*int}(A) \cup \text{gbsb*int}(B) \subseteq \text{gbsb*int}(A \cup B) \).
(ii) \( \text{gbsb*int}(A \cap B) \subseteq \text{gbsb*int}(A) \cap \text{gbsb*int}(B) \).

Proof:
(i) Let \( A \) and \( B \) be subsets of \( X \). By Theorem 3.5(iii), \( \text{gbsb*int}(A) \subseteq \text{gbsb*int}(A \cup B) \) and \( \text{gbsb*int}(B) \subseteq \text{gbsb*int}(A \cup B) \) which implies, \( \text{gbsb*int}(A) \cup \text{gbsb*int}(B) \subseteq \text{gbsb*int}(A \cup B) \).
(ii) By Theorem 3.5(iii), \( \text{gbsb*int}(A \cap B) \subseteq \text{gbsb*int}(A) \) and \( \text{gbsb*int}(A \cap B) \subseteq \text{gbsb*int}(B) \) which implies \( \text{gbsb*int}(A \cap B) \subseteq \text{gbsb*int}(A) \cap \text{gbsb*int}(B) \).

Theorem 3.7. For any subset \( A \) of \( X \),
(i) \( \text{int}(\text{gbsb*int}(A)) = \text{int}(A) \)
(ii) \( \text{gbsb*int}(\text{int}(A)) = \text{int}(A) \).

Proof: (i) Since \( \text{gbsb*int}(A) \subseteq A \), \( \text{int}(\text{gbsb*int}(A)) \subseteq \text{int}(A) \). By Theorem 3.5(iv), \( \text{int}(A) \subseteq (\text{gbsb*int}(A)) \) and so \( \text{int}(A) \subseteq (\text{gbsb*int}(A)) \cap (\text{gbsb*cl}(A)) \). Hence \( \text{int}(\text{gbsb*int}(A)) = \text{int}(A) \).
(ii) Since \( \text{int}(A) \) is open and hence gbsb*-open, by Theorem 3.4(ii), \( \text{gbsb*int}(\text{int}(A)) = \text{int}(A) \).

The generalized b-strongly b*-closure operator

Definition 4.1. Let \( A \) be a subset of a topological space \( (X, \tau) \). Then the intersection of all gbsb*-closed sets in \( X \) containing \( A \) is called the gbsb*-closure of \( A \) and it is denoted by \( \text{gbsb*cl}(A) \). That is, \( \text{gbsb*cl}(A) = \bigcap \{ F : A \subseteq F \text{ and } F \in \text{gbsb*-C}(X) \} \).

Remark 4.2. Since the intersection of gbsb*-closed set is gbsb*-closed, \( \text{gbsb*cl}(A) \) is gbsb*-closed.

Theorem 4.3. Let \( A \) be a subset of a topological space \( (X, \tau) \). Then
(i) \( \text{gbsb*cl}(A) \) is the smallest gbsb*-closed set containing \( A \).
(ii) \( A \) is gbsb*-closed if and only if \( \text{gbsb*cl}(A) = A \).

Theorem 4.4. Let \( A \) and \( B \) be two subsets of a topological space \( (X, \tau) \). Then the following results hold.
(i) \( \text{gbsb*cl}(\emptyset) = \emptyset \) and \( \text{gbsb*cl}(X) = X \).
(ii) If \( B \) is any gbsb*-closed set containing \( A \), then \( \text{gbsb*cl}(A) \subseteq B \).
(iii) If \( A \subseteq B \), then \( \text{gbsb*cl}(A) \subseteq \text{gbsb*cl}(B) \).
(iv) \( A \subseteq \text{gb-cl}(A) \subseteq \text{gbsb*cl}(A) \subseteq \text{bcl}(A) \).
(v) \( \text{gbsb*cl}(\text{gbsb*cl}(A)) = \text{gbsb*cl}(A) \).

Theorem 4.5. Let \( A \) and \( B \) be subsets of a topological space \( (X, \tau) \). Then,
(i) \( \text{gbsb*cl}(A) \cup \text{gbsb*cl}(B) \subseteq \text{gbsb*cl}(A \cup B) \).
(ii) \( \text{gbsb*cl}(A \cap B) \subseteq \text{gbsb*cl}(A) \cap \text{gbsb*cl}(B) \).

Theorem 4.6. For a subset \( A \) of \( X \) and \( x \in X \), \( x \in \text{gbsb*cl}(A) \) if and only if \( \forall A \neq \emptyset \) for every gbsb*-open set \( V \) containing \( x \).

Proof: Necessity: Suppose \( x \in \text{gbsb*cl}(A) \). If there is a gbsb*-open set \( V \) containing \( x \) such that \( \forall A = \emptyset \), then \( A \subseteq X \setminus V \) and \( X \setminus V \) is gbsb*-closed and hence \( \text{gbsb*cl}(A) \subseteq X \setminus V \). Since \( x \in \text{gbsb*cl}(A) \), then \( x \in X \setminus V \) which contradicts to \( x \in V \).
Sufficiency: Assume that $\forall A \neq \emptyset$ for every gbsb*-open set $V$ containing $x$. If $x \notin \text{gbsb*}cl(A)$, then there exists a gbsb*-closed set $F$ such that $A \subseteq F$ and $x \notin F$. Therefore $x \in X \setminus F$, $A \cap (X \setminus F) = \emptyset$ and $X \setminus F$ is gbsb*-open. This is a contradiction to our assumption. Hence $x \in \text{gbsb*}cl(A)$.

**Theorem 4.7.** For any subset $A$ of $X$,

(i) $\text{cl}(\text{gbsb*}cl(A)) = \text{cl}(A)$

(ii) $\text{gbsb*}cl(\text{cl}(A)) = \text{cl}(A)$.

**Generalized b-strongly b*-derived set**

**Definition 5.1.** Let $A$ be a subset of a topological space $X$. A point $x \in X$ is said to be a gbsb*-limit point of $A$ if $\text{G}(A \setminus \{x\}) = \emptyset$, equivalently $x \in \text{G}(A)$ and $\text{G}(A \setminus \{x\}) = \emptyset$, for every gbsb*-open set $G$ containing $x$.

**Definition 5.2.** The set of all gbsb*-limit points of $A$ is called a gbsb*-derived set of $A$ and is denoted by $D_{\text{gbsb}}(A)$.

**Remark 5.3.** For a subset $A$ of $X$, a point $x \in X$ is not a gbsb*-limit point of $A$ if and only if there exists a gbsb*-open set $G$ in $X$ such that $\text{G}(A \setminus \{x\}) = \emptyset$, equivalently $x \in \text{G}(A)$ and $\text{G}(A \setminus \{x\}) = \emptyset$, for every gbsb*-open set $G$ containing $x$.

**Theorem 5.4.** Let $\tau_1$ and $\tau_2$ be topologies on $X$ such that $\text{gbsb*}O(X, \tau_1) \subseteq \text{gbsb*}O(X, \tau_2)$. For any subset $A$ of $X$, every gbsb*-limit point of $A$ with respect to $\text{gbsb*}O(X, \tau_1)$ is a gbsb*-limit point of $A$ with respect to $\text{gbsb*}O(X, \tau_2)$.

**Proof:** Let $A$ be any subset of $X$ and $x \in X$ be a gbsb*-limit point of $A$ with respect to $\text{gbsb*}O(X, \tau_1)$. Then $\text{G}(A \setminus \{x\}) = \emptyset$, for every gbsb*-open set $G$ in $(X, \tau_2)$ containing $x$. But $\text{gbsb*}O(X, \tau_2) \subseteq \text{gbsb*}O(X, \tau_1)$, so in particular $\text{G}(A \setminus \{x\}) = \emptyset$, for every gbsb*-open set $G$ in $(X, \tau_1)$ containing $x$. Hence $x$ is a gbsb*-limit point of $A$ with respect to $\text{gbsb*}O(X, \tau_1)$.

**Theorem 5.5.** Let $A$ and $B$ be subsets of $(X, \tau)$. Then the following are valid:

(i) $D_{\text{gbsb}}(\emptyset) = \emptyset$

(ii) $x \notin D_{\text{gbsb}}(A)$ implies $x \notin D_{\text{gbsb}}(A \setminus \{x\})$

(iii) If $A \subseteq B$, then $D_{\text{gbsb}}(A) \subseteq D_{\text{gbsb}}(B)$

(iv) $D_{\text{gbsb}}(A) \cup D_{\text{gbsb}}(B) \subseteq D_{\text{gbsb}}(A \cup B)$ and $D_{\text{gbsb}}(A \cap B) \subseteq D_{\text{gbsb}}(A) \cap D_{\text{gbsb}}(B)$

(v) $D_{\text{gbsb}}(D_{\text{gbsb}}(A)) = D_{\text{gbsb}}(A)$

(vi) $D_{\text{gbsb}}(A \cup D_{\text{gbsb}}(B)) \subseteq D_{\text{gbsb}}(A \cup B)$

**Proof:**

(i) Obviously $D_{\text{gbsb}}(\emptyset) = \emptyset$. Let $x \in D_{\text{gbsb}}(A)$. Then $x$ is a gbsb*-limit point of $A$. That is, every gbsb*-open set containing $x$ contains at least one point of $A$ other than $x$. Therefore $x$ is a gbsb*-limit point of $A \setminus \{x\}$. This proves (ii).

(iii) Suppose $A \subseteq B$. Let $x \notin D_{\text{gbsb}}(A)$. Then $\text{G}(A \setminus \{x\}) = \emptyset$, for every gbsb*-open set $G$ in $(X, \tau)$ containing $x$. Since $A \subseteq B$, $A \setminus \{x\} \subseteq B \setminus \{x\}$ and hence $\text{G}(B \setminus \{x\}) = \emptyset$. Therefore $x \notin D_{\text{gbsb}}(B)$. This proves (iii).

(iv) Let $A$ and $B$ be subsets of $X$. By part (iii), $D_{\text{gbsb}}(A) \subseteq D_{\text{gbsb}}(A \cup B)$ and $D_{\text{gbsb}}(B) \subseteq D_{\text{gbsb}}(A \cup B)$ which implies that, $D_{\text{gbsb}}(A) \cup D_{\text{gbsb}}(B) \subseteq D_{\text{gbsb}}(A \cup B)$. Again by part (iii), $D_{\text{gbsb}}(A \cap B) \subseteq D_{\text{gbsb}}(A) \cap D_{\text{gbsb}}(B)$ which implies $D_{\text{gbsb}}(A \cup B) \subseteq D_{\text{gbsb}}(A \cap B)$. This proves (iii).

(v) Let $x \in D_{\text{gbsb}}(A)$. Then $\text{G}(A \setminus \{x\}) = \emptyset$, for every gbsb*-open set $G$ in $(X, \tau)$ containing $x$. Let $y \notin \text{G}(A \setminus \{x\})$. This implies $y \in G$ and $y \notin D_{\text{gbsb}}(A)$ with $y \neq x$ and hence $\text{G}(A \setminus \{y\}) \neq \emptyset$. Take $z \in \text{G}(A \setminus \{y\})$, then $x \neq z$ because $x \notin A$. Hence $\text{G}(A \setminus \{x\}) \neq \emptyset$ and therefore $x \notin D_{\text{gbsb}}(A)$. This proves (v).

(vi) Let $x \in D_{\text{gbsb}}(A \cup D_{\text{gbsb}}(B))$. If $x \in A$, then the result is obvious. Assume $x \notin A$. Since $x \in D_{\text{gbsb}}(A \cup D_{\text{gbsb}}(B))$, then $\text{G}(A \cup D_{\text{gbsb}}(B)) \setminus \{x\} = \emptyset$, for every gbsb*-open set $G$ containing $x$. Hence either $G \cap (A \setminus \{x\}) = \emptyset$ or $G \cap (D_{\text{gbsb}}(B) \setminus \{x\}) = \emptyset$. If $G \cap (A \setminus \{x\}) = \emptyset$, then $x \notin D_{\text{gbsb}}(A)$. If $G \cap (D_{\text{gbsb}}(B) \setminus \{x\}) = \emptyset$, then $x \notin D_{\text{gbsb}}(B)$. Since $x \notin A$ gives that $x \notin D_{\text{gbsb}}(A \cup D_{\text{gbsb}}(B))$ and by part (v), $x \in D_{\text{gbsb}}(A)$. Hence $D_{\text{gbsb}}(A \cup D_{\text{gbsb}}(B)) \subseteq D_{\text{gbsb}}(A)$

**Theorem 5.6.** For any subset $A$ of $X$, we have $D_{\text{gbsb}}(A) \subseteq \text{gbsb*}cl(A)$.

**Proof:** Let $x \notin D_{\text{gbsb}}(A)$. Then $\text{G}(A \setminus \{x\}) = \emptyset$, for every gbsb*-open set $G$ in $(X, \tau)$ containing $x$. That is, $G \cap A = \emptyset$, for every gbsb*-open set $G$ in $(X, \tau)$ containing $x$. By Theorem 4.6, $x \notin \text{gbsb*}cl(A)$. Hence $D_{\text{gbsb}}(A) \subseteq \text{gbsb*}cl(A)$.
Theorem 5.7. Let A be a subset of X. Then $gbsb^*(c)(A)=A \cup D_{gbsb^*}(A)$.
Proof: Let $x \in gbsb^*(c)(A)$. If $x \in A$, then the result is obvious. Suppose $x \notin A$. Since $x \in gbsb^*(c)(A)$, then by Theorem 4.6, $G \cap A = \emptyset$, for every $gbsb^*$-open set $G$ in $(X, \tau)$ containing $x$. Since $x \notin A$, then we have $G \cap (A\setminus x) = \emptyset$ and therefore $x \in D_{gbsb^*}(A)$. Hence $gbsb^*(c)(A) \subseteq A \cup D_{gbsb^*}(A)$. On the other hand, we know that $A \cap gbsb^*(c)(A)$ and by Theorem 5.6, $D_{gbsb^*}(A) \subseteq gbsb^*(c)(A)$, then we conclude that $A \cup D_{gbsb^*}(A) \subseteq gbsb^*(c)(A)$. Hence $gbsb^*(c)(A)=A \cup D_{gbsb^*}(A)$.

Theorem 5.8. Let A and B be subsets of X. If $A \in gbsb^*-O(X)$ and $gbsb^*-O(X)$ is topology on X, then $A \cap gbsb^*(c)(B) \subseteq gbsb^*(c)(A \cap B)$.
Proof: Suppose $A \in gbsb^*-O(X)$ and $gbsb^*-O(X)$ is topology on X. Let $x \in A \cap gbsb^*(c)(B)$. Then $x \in A$ and $x \in gbsb^*(c)(B)$.
By Theorem 5.7, $x \in B \cup D_{gbsb^*}(B)$. Then we have two cases,

a) If $x \in B$, then $x \in A \cap B \subseteq gbsb^*(c)(A \cap B)$,

b) If $x \notin B$, then $x \in D_{gbsb^*}(B)$ and so $G \cap B = \emptyset$, for every $gbsb^*$-open set $G$ in $(X, \tau)$ containing $x$. Since $A, G \in gbsb^*-O(X)$ and $gbsb^*-O(X)$ is topology on X, then $A \cap G$ is also a $gbsb^*$-open set containing $x$. Hence $G \cap (A \cap B)=(G \cap A) \cap B \neq \emptyset$. That implies, $x \in D_{gbsb^*}(A \cap B)$. By Theorem 5.6, $x \in gbsb^*(c)(A \cap B)$
Hence $A \cap gbsb^*(c)(B) \subseteq gbsb^*(c)(A \cap B)$.

Remark 5.9. If $gbsb^*-O(X, \tau)$ is not a topology on X, then the above theorem need not be true which is shown in the following example.

Example 5.10. Let $X=\{a, b, c, d\}$ with a topology $\tau=\{\emptyset, \{a\}, \{a, b\}, \{a, b, c, d\}, X\}$. Here $gbsb^*-O(X, \tau)=\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $gbsb^*-O(X, \tau)$ is not a topology on X. Let $A=\{a, b, d\}$ and $B=\{a, c, d\}$. Then $A \cap gbsb^*(c)(B)=\{a, b, d\}$ and $gbsb^*(c)(A \cap B)=\{a, d\}$. Hence $A \cap gbsb^*(c)(B) \subseteq gbsb^*(c)(A \cap B)$.

Theorem 5.11. If A is a subset of a discrete topological space X, then $D_{gbsb^*}(A)=\emptyset$.
Proof: Let A be a subset of a discrete topological space X and x be any element of X. Since X is a discrete topology, every subset of X is open and so $gbsb^*$-open. In particular the singleton set $G=\{x\}$ is $gbsb^*$-open and therefore $G \cap (A \setminus x) = \emptyset$. Then we conclude that x is not a $gbsb^*$-limit point of A. Since x\in X is arbitrary, then every element of X is not a $gbsb^*$-limit point of A. Hence $D_{gbsb^*}(A)=\emptyset$.

Theorem 5.12. For any subset A of X, $gbsb^*(c)(A)=A \cup D_{gbsb^*}(X \setminus A)$.
Proof: Let $x \in gbsb^*(c)(A)$. Then there exists a $gbsb^*$-open set G such that $x \in G \subseteq \setminus A$. That implies, $G \cap (X \setminus A) = \emptyset$ where G is a $gbsb^*$-open set containing x and hence $x \in D_{gbsb^*}(X \setminus A)$. Therefore $x \in A \cup D_{gbsb^*}(X \setminus A)$. Hence $gbsb^*(c)(A) \subseteq A \cup D_{gbsb^*}(X \setminus A)$. Let $x \in A \cup D_{gbsb^*}(X \setminus A)$. Then $x \in D_{gbsb^*}(X \setminus A)$ and therefore there exists a $gbsb^*$-open set G containing x such that $G \cap (X \setminus A) = \emptyset$. That is, x \in G \subseteq \setminus A. Hence x is a $gbsb^*$-interior point of A. Therefore $x \in gbsb^*(c)(A)$ and so $A \cup D_{gbsb^*}(X \setminus A) \subseteq gbsb^*(c)(A)$. Hence $gbsb^*(c)(A)=A \cup D_{gbsb^*}(X \setminus A)$.

Theorem 5.13. For any subset A of X,
(i) $X \cap gbsb^*(c)(A)=gbsb^*(c)(X \setminus A)$
(ii) $X \cap gbsb^*(c)(X \setminus A)=gbsb^*(c)(A)$
(iii) $X \cap gbsb^*(c)(A)=gbsb^*(c)(X \setminus A)$
(iv) $X \cap gbsb^*(c)(A)=gbsb^*(c)(X \setminus A)$

Proof. (i) $X \cap gbsb^*(c)(A) = X \cap (A \cup D_{gbsb^*}(X \setminus A)) = (X \setminus A) \cup D_{gbsb^*}(X \setminus A)=gbsb^*(c)(X \setminus A)$.
(ii) Replace A by XA in part(i), $X \cap gbsb^*(c)(X \setminus A)=gbsb^*(c)(A)$.
(iii) $gbsb^*(c)(X \setminus A) \cap D_{gbsb^*}(X \setminus A) = X \cap D_{gbsb^*}(X \setminus A) = X \cap gbsb^*(c)(X \setminus A)$. This proves (iii).
(iv) Replace A by XA in part(iii), $X \cap gbsb^*(c)(X \setminus A)=gbsb^*(c)(X \setminus A)$.

gbsb^*-border and gbsb^*-frontier of a set
Definition 6.1. Let A be a subset of X. Then the set $B_{gbsb^*}(A)=A \setminus gbsb^*(c)(A)$ is called the gbsb^*-border of A and the set $F_{gbsb^*}(A)=gbsb^*(c)(A) \setminus gbsb^*(c)(A)$ is called the gbsb^*-frontier of A.

Example 6.2. Let $X=\{a, b, c, d\}$ with a topology $\tau=\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$. Here $gbsb^*-O(X)=\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Let $A=\{a, b, c\}$. Then $gbsb^*(c)(A)=X$ and

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gbsb*int(A)={a, b, c}. Therefore B_{gbsb*}(A)=A\gbsb*int(A)={a, b, c}\{a, b, c\}=\emptyset and Fr_{gbsb*}(A)=gbsb*cl(A)\gbsb*int(A)=X\{a, b, c\}=\{d\}.

**Theorem 6.3.** If a subset A of X is gbsb*-closed, then B_{gbsb*}(A)=Fr_{gbsb*}(A).
Proof: Let A be a gbsb*-closed subset of X. Then by Theorem 4.3, gbsb*cl(A)=A. Now, Fr_{gbsb*}(A)=gbsb*cl(A)\gbsb*int(A)=A\gbsb*int(A)=B_{gbsb*}(A).

**Theorem 6.4.** For a subset A of X, the following statements are hold:
(i) A=gbsb*int(A)UB_{gbsb*}(A)
(ii) gbsb*int(A)\cap B_{gbsb*}(A)=\emptyset
(iii) A is gbsb*-open if and only if B_{gbsb*}(A)=\emptyset
(iv) B_{gbsb*}(gbsb*int(A))=\emptyset
(v) gbsb*int(B_{gbsb*}(A))=\emptyset
(vi) B_{gbsb*}(B_{gbsb*}(A))=B_{gbsb*}(A)

Proof:
(i) Let x\in A. If x\in gbsb*int(A), then the result is obvious. If x\notin gbsb*int(A), then by the definition of B_{gbsb*}(A), x \in B_{gbsb*}(A). Hence x \in gbsb*int(A) \cup B_{gbsb*}(A) and so A \subseteq gbsb*int(A) \cup B_{gbsb*}(A). On the other hand, since gbsb*int(A) \subseteq A and B_{gbsb*}(A) \subseteq A, then we have gbsb*int(A) \cup B_{gbsb*}(A) \subseteq A. This proves (i).
(ii) Suppose gbsb*int(A)\cap B_{gbsb*}(A)=\emptyset. Then x \notin gbsb*int(A) and x \notin B_{gbsb*}(A). Since B_{gbsb*}(A)=A\gbsb*int(A), then x \in A but x \notin gbsb*int(A), x \notin A. There is a contradiction. Hence gbsb*int(A)\cap B_{gbsb*}(A)=\emptyset.
(iii) Necessity: Suppose A is gbsb*-open. Then by Theorem 3.4, gbsb*int(A)=A. Now, B_{gbsb*}(A)=A\gbsb*int(A)=A=\emptyset. Sufficiency: Suppose B_{gbsb*}(A)=\emptyset. This implies, A\subseteq gbsb*int(A)=\emptyset. Therefore A=gbsb*int(A) and hence A is gbsb*-open.
(iv) By the definition of gbsb* border of a set, B_{gbsb*}(gbsb*int(A))=gbsb*int(A)\gbsb*int(gbsb*int(A)). By Theorem 3.4, gbsb*int(gbsb*int(A))=gbsb*int(A) and hence B_{gbsb*}(gbsb*int(A))=\emptyset.
(v) Let x \in gbsb*int(B_{gbsb*}(A)). Since B_{gbsb*}(A) \subseteq A, by Theorem 3.5, gbsb*int(B_{gbsb*}(A)) \subseteq gbsb*int(A). Hence x \notin gbsb*int(A). Since gbsb*int(B_{gbsb*}(A)) \subseteq gbsb*int(A), then x \in B_{gbsb*}(A). Therefore x \in gbsb*int(A) \cap B_{gbsb*}(A). By part (ii), x=\emptyset. This proves (v).
(vi) By the definition of gbsb* border of a set, B_{gbsb*}(B_{gbsb*}(A))=B_{gbsb*}(A)\gbsb*int(B_{gbsb*}(A)). By part (v), gbsb*int(B_{gbsb*}(A))=\emptyset and hence B_{gbsb*}(B_{gbsb*}(A))=B_{gbsb*}(A).

**Theorem 6.5.** Let A be a subset of X. Then,
(i) B_{gbsb*}(A)=A\cap gbsb*cl(XA)
(ii) B_{gbsb*}(A)=A\cap D_{gbsb*}(XA).

Proof:
(i) Since B_{gbsb*}(A)=A\gbsb*int(A) and by Theorem 5.13, B_{gbsb*}(A)=A\cap gbsb*cl(XA)
=\emptyset \cap (X\cap gbsb*cl(XA))=\emptyset \cap gbsb*cl(XA).
(ii) By Theorem 6.5 and Theorem 5.7, we have B_{gbsb*}(A)=A \cap gbsb*cl(XA)=A \cap (XA) \cup D_{gbsb*}(XA)=A \cap (X \cap gbsb*cl(XA)) \cup D_{gbsb*}(X \cap gbsb*cl(XA))=A \cap (X \cap gbsb*cl(XA)) \cup D_{gbsb*}(X \cap gbsb*cl(XA)).

**Theorem 6.6.** Let A be a subset of X. Then A is gbsb*-closed if and only if Fr_{gbsb*}(A) \subseteq A.
Proof: Necessity: Suppose A is gbsb*-closed. Then by Theorem 4.3, gbsb*cl(A)=A. Now, Fr_{gbsb*}(A)=gbsb*cl(A)\gbsb*int(A)=A\gbsb*int(A) \subseteq A. Hence Fr_{gbsb*}(A) \subseteq A. Sufficiency: Assume that, Fr_{gbsb*}(A) \subseteq A. Then gbsb*cl(A)\gbsb*int(A) \subseteq A. Since gbsb*int(A) \subseteq A, then we conclude that gbsb*cl(A) \subseteq A. Also A \subseteq gbsb*cl(A). Therefore gbsb*cl(A) \subseteq A and hence A is gbsb*-closed.

**Theorem 6.7.** For a subset A of X, we have the following
(i) gbsb*cl(A)=\emptyset \cup Fr_{gbsb*}(A)
(ii) gbsb*int(A) \cap Fr_{gbsb*}(A)=\emptyset
(iii) B_{gbsb*}(A) \subseteq Fr_{gbsb*}(A)
(iv) Fr_{gbsb*}(A)=B_{gbsb*}(A) \cup D_{gbsb*}(A)\gbsb*int(A).
(v) If A is gbsb*-open then Fr_{gbsb*}(A)=B_{gbsb*}(A).

(vi) \( \text{Fr}_{gbsh}(A)=gbsb^*\text{cl}(A)\cap gbsb^*\text{cl}(X\setminus A) \).

Proof:
(i) Since \( gbsb^*\text{int}(A)\subseteq gbsb^*\text{cl}(A) \) and \( \text{Fr}_{gbsh}(A)\subseteq gbsb^*\text{cl}(A) \), then \( gbsb^*\text{int}(A) \cup gbsb^*\text{cl}(A) \subseteq gbsb^*\text{cl}(A) \). Let \( x \in gbsb^*\text{cl}(A) \). Suppose \( x \in \text{Fr}_{gbsh}(A) \). Since, then \( x \in gbsb^*\text{int}(A) \). Hence \( x \in gbsb^*\text{int}(A) \cup gbsb^*\text{cl}(A) \) and hence \( gbsb^*\text{cl}(A) \subseteq gbsb^*\text{int}(A) \cup gbsb^*\text{cl}(A) \). This proves (i).
(ii) Suppose \( gbsb^*\text{int}(A) \cap \text{Fr}_{gbsh}(A) \neq \emptyset \). Let \( x \in gbsb^*\text{int}(A) \cap \text{Fr}_{gbsh}(A) \). Then \( x \in gbsb^*\text{cl}(A) \) since \( \text{Fr}_{gbsh}(A) = gbsb^*\text{cl}(A) \) and \( x \in gbsb^*\text{int}(A) \), which is impossible to \( x \) belongs to both \( gbsb^*\text{cl}(A) \) and \( \text{Fr}_{gbsh}(A) \), since \( \text{Fr}_{gbsh}(A) = gbsb^*\text{cl}(A) \). Hence \( gbsb^*\text{int}(A) \cap \text{Fr}_{gbsh}(A) = \emptyset \).
(iii) Since \( A \subseteq gbsb^*\text{cl}(A) \), then \( A \cap gbsb^*\text{int}(A) \subseteq gbsb^*\text{cl}(A) \cap gbsb^*\text{int}(A) \). That implies, \( B_{gbsh}(A) \subseteq \text{Fr}_{gbsh}(A) \).
(iv) Since \( \text{Fr}_{gbsh}(A) = gbsb^*\text{cl}(A) \cap gbsb^*\text{int}(A) \) and by Theorem 5.7, \( \text{Fr}_{gbsh}(A) = (A \cup D_{gbsh}(A)) \cap gbsb^*\text{int}(A) = (A \cup D_{gbsh}(A)) \cap (B_{gbsh}(A) \cup D_{gbsh}(A)) \cap gbsb^*\text{int}(A) \). This proves (iv).
(v) Assume that \( A \) is \( gbsb^* \)-open. Then by Theorem 3.4, \( gbsb^*\text{int}(A) = \emptyset \) and by Theorem 6.4, \( B_{gbsh}(A) = \emptyset \). By part (iv), \( \text{Fr}_{gbsh}(A) = B_{gbsh}(A) \cup (D_{gbsh}(A) \cap gbsb^*\text{int}(A)) = \emptyset \cup (D_{gbsh}(A) \cap \emptyset) = D_{gbsh}(A) \cap \emptyset = \emptyset \). Then by Theorem 6.5, \( \text{Fr}_{gbsh}(A) = B_{gbsh}(X \setminus A) \).
(vi) Since \( gbsb^*\text{cl}(X \setminus A) = \text{Fr}_{gbsh}(A) \), \( gbsb^*\text{cl}(A) \cap gbsb^*\text{cl}(X \setminus A) = gbsb^*\text{cl}(A) \cap \text{Fr}_{gbsh}(A) = gbsb^*\text{cl}(A) \cap \text{Fr}_{gbsh}(A) = \text{Fr}_{gbsh}(A) \).

**Theorem 6.8.** For a subset \( A \) of \( X \), we have the following

(i) \( \text{Fr}_{gbsh}(A) = \text{Fr}_{gbsh}(X \setminus A) \).
(ii) \( \text{Fr}_{gbsh}(A) = gbsb^*\text{closed} \).
(iii) \( \text{Fr}_{gbsh}(A) \subseteq gbsb^*\text{cl}(A) \).
(iv) \( \text{Fr}_{gbsh}(A) \cap gbsb^*\text{int}(A) \subseteq \text{Fr}_{gbsh}(A) \).
(v) \( \text{Fr}_{gbsh}(A) = gbsb^*\text{cl}(A) \).
(vi) \( gbsb^*\text{int}(A) \subseteq \text{Fr}_{gbsh}(A) \).

Proof:
(i) By Theorem 6.7 (vi), \( \text{Fr}_{gbsh}(A) = gbsb^*\text{cl}(A) \cap gbsb^*\text{cl}(X \setminus A) = \text{Fr}_{gbsh}(X \setminus A) \).
(ii) Now, \( gbsb^*\text{cl}(\text{Fr}_{gbsh}(A)) = gbsb^*\text{cl}(gbsb^*\text{cl}(A) \cap gbsb^*\text{cl}(X \setminus A)) = gbsb^*\text{cl}(A) \cap gbsb^*\text{cl}(X \setminus A) = \text{Fr}_{gbsh}(A) \). That is, \( gbsb^*\text{cl}(\text{Fr}_{gbsh}(A)) = \text{Fr}_{gbsh}(A) \). Also \( \text{Fr}_{gbsh}(A) \subseteq gbsb^*\text{Fr}_{gbsh}(A) \). Therefore \( gbsb^*\text{cl}(\text{Fr}_{gbsh}(A)) = \text{Fr}_{gbsh}(A) \) and hence \( \text{Fr}_{gbsh}(A) = gbsb^*\text{closed} \).
(iii) By part (ii), \( \text{Fr}_{gbsh}(A) = gbsb^*\text{cl}(A) \cap gbsb^*\text{cl}(X \setminus A) = \text{Fr}_{gbsh}(X \setminus A) \).
(iv) By the definition of \( gbsb^* \)-frontier, \( \text{Fr}_{gbsh}(gbsb^*\text{int}(A)) = gbsb^*\text{cl}(gbsb^*\text{int}(A)) \cap gbsb^*\text{cl}(A) \) and hence \( \text{Fr}_{gbsh}(gbsb^*\text{int}(A)) = gbsb^*\text{cl}(A) \). Hence \( \text{Fr}_{gbsh}(gbsb^*\text{int}(A)) \subseteq \text{Fr}_{gbsh}(A) \).
(v) By the definition of \( gbsb^* \)-frontier, \( \text{Fr}_{gbsh}(gbsb^*\text{cl}(A)) = gbsb^*\text{cl}(gbsb^*\text{cl}(A)) \cap gbsb^*\text{cl}(A) \) and hence \( \text{Fr}_{gbsh}(gbsb^*\text{cl}(A)) \subseteq \text{Fr}_{gbsh}(A) \).
(vi) Now \( A \subseteq \text{Fr}_{gbsh}(A) \) and \( A \subseteq gbsb^*\text{int}(A) \).

**gbsb^*-exterior and gbsb^*-kernel**

**Definition 7.1.** Let \( A \) be a subset of a topological space \( (X, \tau) \). Then the \( gbsb^* \)-interior of \( X \setminus A \) is called the \( gbsb^* \)-exterior of \( A \) and it is denoted by \( \text{Ext}_{gbsh}(A) \). That is, \( \text{Ext}_{gbsh}(A) = gbsb^*\text{int}(X \setminus A) \).

**Example 7.2.** Let \( X = \{a, b, c, d\} \) with a topology \( \tau = \{\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\} \). Here \( gbsb^*\text{O}(X) = \{\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\} \). Let \( A = \{c, d\} \). Then \( \text{Ext}_{gbsh}(A) = gbsb^*\text{int}(X \setminus A) = \{a, b\} \).

**Theorem 7.3.** For a subsets \( A \) and \( B \) of \( X \), the following are valid.

(i) \( \text{Ext}_{gbsh}(A) = X \setminus gbsb^*\text{cl}(A) \).
(ii) \( \text{Ext}_{gbsh}(A \cup B) = gbsb^*\text{int}(gbsb^*\text{cl}(A) \cup gbsb^*\text{cl}(B)) \).
(iii) If \( A \subseteq B \), then \( \text{Ext}_{gbsh}(B) \subseteq \text{Ext}_{gbsh}(A) \).
(iv) \( \text{Ext}_{gbsh}(A \setminus B) \subseteq \text{Ext}_{gbsh}(A) \setminus \text{Ext}_{gbsh}(B) \).
(v) \( \text{Ext}_{gbsh}(A \cup B) \subseteq \text{Ext}_{gbsh}(A) \cup \text{Ext}_{gbsh}(B) \).
(vi) \( \text{Ext}_{gbsh}(\emptyset) = \emptyset \) and \( \text{Ext}_{gbsh}(X) = X \).
(vii) \( \text{Ext}_{gbsh}(A) = \text{Ext}_{gbsh}(X \setminus \text{Ext}_{gbsh}(A)) \).

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Proof.
(i) Since, \( X \cap \text{gbsb}\ast\text{cl}(A) = \text{gbsb}\ast\text{int}(X\setminus A) \), Ext_{gbsb}\ast(A) = X \cap \text{gbsb}\ast\text{cl}(A).
(ii) Ext_{gbsb}\ast(\text{Ext}_{gbsb}\ast(A)) = Ext_{gbsb}\ast(\text{gbsb}\ast\text{int}(X\setminus A)) = \text{gbsb}\ast\text{int}(\text{gbsb}\ast\text{cl}(A)) = \text{gbsb}\ast\text{int}(A).
(iii) Suppose \( A \subseteq B \). Then, Ext_{gbsb}\ast(B) = \text{gbsb}\ast\text{int}(X\setminus B) \subseteq \text{gbsb}\ast\text{int}(X\setminus A) = \text{Ext}_{gbsb}\ast(A).
(iv) Ext_{gbsb}\ast(A \cup B) = \text{gbsb}\ast\text{int}(X(A \cup B)) = \text{gbsb}\ast\text{int}((X\setminus A) \cap (X\setminus B)) \subseteq \text{gbsb}\ast(X\setminus A) \cap \text{gbsb}\ast\text{cl}(X\setminus B) = \text{Ext}_{gbsb}\ast(A) \cap \text{Ext}_{gbsb}\ast(B).
(v) \text{Ext}_{gbsb}\ast((A \setminus B)) = \text{gbsb}\ast\text{int}(X(A \setminus B)) = \text{gbsb}\ast\text{int}(X(A \setminus B)) \supseteq \text{gbsb}\ast(X\setminus A) \cap \text{gbsb}\ast\text{cl}(X\setminus B) = \text{Ext}_{gbsb}\ast(A) \cup \text{Ext}_{gbsb}\ast(B).
(vi) \text{Ext}_{gbsb}\ast(X) = \text{gbsb}\ast\text{int}(X \setminus A) = \text{gbsb}\ast\text{int}(A \setminus B) = \text{gbsb}\ast\text{int}(X \setminus A) = \text{gbsb}\ast\text{int}(X \setminus A) = \text{gbsb}\ast\text{int}(X \setminus A).

Definition 7.4. Let \( A \) be a subset of a topological space \( X \). Then the intersection of all gbsb\ast-open sets containing \( A \) is called the gbsb\ast-kernel of \( A \). It is denoted by \( \text{gbsb}\ast\text{ker}(A) \). That is, \( \text{gbsb}\ast\text{ker}(A) = \cap \{ U \in \text{gbsb}\ast\text{O}(X, \tau) \text{ and } A \subseteq U \} \).

Theorem 7.5. Let \( A \) and \( B \) be subsets of \( (X, \tau) \). Then the following results hold.
(i) \( A \subseteq \text{gbsb}\ast\text{ker}(A) \).
(ii) If \( A \subseteq B \), then \( \text{gbsb}\ast\text{ker}(A) \subseteq \text{gbsb}\ast\text{ker}(B) \).
(iii) If \( A \) is gbsb\ast-open, then \( \text{gbsb}\ast\text{ker}(A) = A \).
(iv) \( \text{gbsb}\ast\text{ker}(\text{gbsb}\ast\text{ker}(A)) = \text{gbsb}\ast\text{ker}(A) \).

Proof:
(i) Since \( \text{gbsb}\ast\text{ker}(A) \) is the intersection of all gbsb\ast-open sets containing \( A \), then we have \( A \subseteq \text{gbsb}\ast\text{ker}(A) \).
(ii) Suppose \( A \subseteq B \). Let \( U \) be any gbsb\ast-open set containing \( B \), since \( A \subseteq B \), then \( A \subseteq U \) and hence by the definition of \( \text{gbsb}\ast\text{ker}(A) \), \( \text{gbsb}\ast\text{ker}(A) \subseteq U \). Therefore, \( \text{gbsb}\ast\text{ker}(A) \subseteq \{ \text{U} \in \text{gbsb}\ast\text{O}(X, \tau) \text{ and } A \subseteq U \} = \text{gbsb}\ast\text{ker}(B) \). This proves (ii).
(iii) Suppose \( A \) is gbsb\ast-open. Then by Definition, \( \text{gbsb}\ast\text{ker}(A) \subseteq A \). But \( A \subseteq \text{gbsb}\ast\text{ker}(A) \) and therefore \( \text{gbsb}\ast\text{ker}(A) = A \).
(iv) By part (i) and (ii), \( \text{gbsb}\ast\text{ker}(A) \subseteq \text{gbsb}\ast\text{ker}(\text{gbsb}\ast\text{ker}(A)) \). If \( x \in \text{gbsb}\ast\text{ker}(A) \), then there exists a gbsb\ast-open set \( U \) such that \( A \subseteq U \) and \( x \in U \). This implies that, \( \text{gbsb}\ast\text{ker}(\text{gbsb}\ast\text{ker}(A)) \subseteq A \), and so we have \( x \in \text{gbsb}\ast\text{ker}(\text{gbsb}\ast\text{ker}(A)) \). Thus \( \text{gbsb}\ast\text{ker}(\text{gbsb}\ast\text{ker}(A)) = \text{gbsb}\ast\text{ker}(A) \).

Theorem 7.7. Let \( (X, \tau) \) be a topological space. Then \( \cap \{ \text{gbsb}\ast\text{cl}(\{x\})/x \in X \} = \emptyset \) if and only if \( \text{gbsb}\ast\text{ker}(\{x\}) \neq X \) for every \( x \in X \).

Proof. Necessity: Suppose that \( \cap \{ \text{gbsb}\ast\text{cl}(\{x\})/x \in X \} = \emptyset \). Suppose there is a point \( y \) in \( X \) such that \( \text{gbsb}\ast\text{ker}(\{y\}) = X \). Let \( x \) be any point of \( X \). Then \( x \in \text{gbsb}\ast\text{ker}(\{y\}) \) and therefore \( x \in V \) for every gbsb\ast-open set \( V \) containing \( y \). That is, every gbsb\ast-closed set containing \( y \) must contain \( y \) and hence \( y \in \text{gbsb}\ast\text{cl}(\{x\}) \) for any \( x \in X \). This implies that \( y \in \cap \{ \text{gbsb}\ast\text{cl}(\{x\}): x \in X \} \). This is a contradiction to our assumption. Hence \( \text{gbsb}\ast\text{ker}(\{x\}) \neq X \) for every \( x \in X \). Sufficiency: Assume that \( \text{gbsb}\ast\text{ker}(\{x\}) \neq X \), for every \( x \in X \). If there exists a point \( y \) in \( X \) such that \( y \in \cap \{ \text{gbsb}\ast\text{cl}(x)/x \in X \} \), then every gbsb\ast-open set containing \( y \) must contain every point of \( X \). This implies, the space \( X \) is the unique gbsb\ast-open set containing \( y \). Hence \( \text{gbsb}\ast\text{ker}(\{y\}) = X \), which is a contradiction. Therefore, \( \cap \{ \text{gbsb}\ast\text{cl}(\{x\}): x \in X \} = \emptyset \).

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