On The Positive Pell Equation $y^2 = 40x^2 + 1$

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Abstract: The binary quadratic equation respected by the positive pellian $y^2 = 40x^2 + 1$ is analysed for its distinct integer solutions. A few interesting among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: Binary quadratic, hyperbola, parabola, integral solutions, pell equation.

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INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where $D$ is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when $D$ takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 40x^2 + 1$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS

Consider the binary quadratic equation

$y^2 = 40x^2 + 1$

The least positive integer solutions $x_0 = 3, y_0 = 19$

The general solution $(x_n, y_n)$ of (1) is given by

where,

$f_n = (19 + 3\sqrt{40})^{n+1} + (19 - 3\sqrt{40})^{n+1}$
$g_n = (19 + 3\sqrt{40})^{n+1} - (19 - 3\sqrt{40})^{n+1}$

where $n = -1,0,1,2,\ldots$

The recurrence relations satisfied by the solutions (2) are given by

$y_{n+2} - 38y_{n+1} + y_n = 0$
$x_{n+2} - 38x_{n+1} + x_n = 0$
Some numerical examples of $x$ & $y$ are satisfying (1) are given in the table below.

<table>
<thead>
<tr>
<th>n</th>
<th>$x_n$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>19</td>
</tr>
<tr>
<td>1</td>
<td>114</td>
<td>721</td>
</tr>
<tr>
<td>2</td>
<td>4329</td>
<td>27379</td>
</tr>
<tr>
<td>3</td>
<td>164388</td>
<td>1039681</td>
</tr>
<tr>
<td>4</td>
<td>6242415</td>
<td>39480499</td>
</tr>
</tbody>
</table>

From the above, we observe some interesting relations among the solutions which are presented below:

1. The $x_n$ values are alternatively odd and even
2. The $y_n$ values are always odd
3. Each of the following expressions is a nasty number
   - $8y_{2n+1} + 12$
   - $\frac{1368y_{2n+2} - 36y_{2n+4} + 36}{3}$
   - $\frac{12y_{2n+2} - 720x_{2n+1} + 228}{19}$
   - $228y_{2n+2} - 1440x_{2n+2} + 12$
   - $\frac{8652y_{2n+2} - 1440x_{2n+3} + 228}{19}$
   - $12y_{2n+3} - 54720x_{2n+1} + 8652$
   - $\frac{721}{228y_{2n+3} - 54720x_{2n+2} + 228}$
   - $12x_{2n+3} - 8652y_{2n+1} + 1368$
   - $\frac{114}{228x_{2n+3} - 8652y_{2n+2} + 36}$
   - $\frac{3}{8652y_{2n+3} - 54720x_{2n+2} + 12}$
   - $12x_{2n+2} - 228x_{2n+1} + 12$

4. Each of the following expressions is a cubical integer:
   - $2y_{3n+2} + 6y_n$
   - $9[2(14y_{3n+3} - 3y_{3n+4}) + 6(y_{n+1} - 3y_{n+2})]$
   - $2(y_{3n+3} - 120x_{3n+2}) + 6(y_{n+1} - 120x_n)$
   - $2(19y_{3n+3} - 120x_{3n+3}) + 6(19y_{n+1} - 120x_{n+1})$
   - $2(721y_{3n+3} - 120x_{3n+4}) + 6(721y_{n+1} - 120x_{n+2})$
   - $\frac{1}{721}[2(y_{3n+4} - 4560x_{3n+2}) + 6(y_{n+2} - 4560x_n)]$
   - $2(y_{3n+4} - 240x_{3n+3}) + 6(y_{n+2} - 240x_{n+1})$
5. Relations among the solutions:

\[
\begin{align*}
&2y_{n+2} = 76y_{n+1} - 2y_n \\
&120x_{n+1} = 19y_{n+1} - y_n \\
&114y_{n+1} = 3y_{n+2} + 3y_n \\
&y_{n+2} = 240x_{n+1} + y_n \\
&19x_{n+2} = 721x_{n+1} + 3y_n \\
&721y_{n+1} = 120x_{n+2} + 19y_n \\
&721y_{n+2} = 4560x_{n+2} + y_n \\
&721x_n = x_{n+2} - 114y_n \\
&120x_{n+1} = y_{n+2} - 19y_{n+1} \\
&19y_n = y_{n+1} - 120x_n \\
&19y_{n+2} = 721(y_{n+1} - 4440x_n)
\end{align*}
\]

\[
\begin{align*}
&19x_{n+1} = x_n + 3y_{n+1} \\
&19x_{n+2} = 19x_n + 114y_{n+1} \\
&x_n = 19x_{n+1} - 3y_{n+1} \\
&19y_{n+2} = y_{n+1} + 120x_{n+2} \\
&19x_n = 19x_{n+2} - 114y_{n+1} \\
&19x_{n+1} = x_{n+2} - 3y_{n+1} \\
&721y_n = y_{n+2} - 4560x_n \\
&721y_{n+1} = 19y_{n+2} - 120x_n \\
&721x_{n+1} = 3y_{n+2} + 19x_n \\
&721x_{n+2} = 114y_{n+2} + x_n \\
&19y_n = 19y_{n+2} - 4560x_{n+1} \\
&19x_{n+2} = x_{n+1} + 3y_{n+2} \\
&3y_n = x_{n+1} - 19x_n \\
&3y_{n+1} = 19x_{n+1} + 341x_n \\
&x_{n+2} = 38x_{n+1} - x_n \\
&6y_{n+1} = x_{n+2} - x_n \\
&3y_n = 19x_{n+2} - 721x_{n+1} \\
&3y_{n+1} = x_{n+2} - 19x_{n+1} \\
&3y_{n+2} = 19x_{n+2} - x_{n+1}
\end{align*}
\]
3x_n = 114x_{n+1} - 3x_{n+2}
2y_n = 2(721y_{n+1} - 4560x_{n+2})
2y_{n+1} = 38y_{n+2} - 240x_{n+2}

REMARKABLE OBSERVATION

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table 1 below:

<table>
<thead>
<tr>
<th>S.no</th>
<th>(X,Y)</th>
<th>Hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(y_{n+1} - y_n, 2y_n)</td>
<td>90Y^2 - X^2 = 360</td>
</tr>
<tr>
<td>2</td>
<td>(y_{n+2} - 721y_n, 2y_n)</td>
<td>129960Y^2 - X^2 = 4</td>
</tr>
<tr>
<td>3</td>
<td>(x_{n+1} - 3y_n, 2y_n)</td>
<td>361Y^2 - 160X^2 = 4</td>
</tr>
<tr>
<td>4</td>
<td>(x_{n+2} - 114y_n, 2y_n)</td>
<td>51984Y^2 - 160X^2 = 2079364</td>
</tr>
<tr>
<td>5</td>
<td>(38y_{n+2} - 1442y_{n+1}, 114y_{n+1} - 3y_{n+2})</td>
<td>160Y^2 - X^2 = 1440</td>
</tr>
<tr>
<td>6</td>
<td>(y_n, y_{n+1} - 120x_n)</td>
<td>Y^2 - 14440X^2 = 361</td>
</tr>
<tr>
<td>7</td>
<td>(19x_{n+2} - 114y_{n+1}, 721y_{n+1} - 120x_{n+2})</td>
<td>Y^2 - 40X^2 = 361</td>
</tr>
<tr>
<td>8</td>
<td>(x_n, y_{n+2} - 4560x_n)</td>
<td>Y^2 - 20793640X^2 = 519841</td>
</tr>
<tr>
<td>9</td>
<td>(721x_{n+1} - 3y_{n+2}, 19y_{n+2} - 4560y_{n+1})</td>
<td>Y^2 - 40X^2 = 361</td>
</tr>
<tr>
<td>10</td>
<td>(721x_{n+2} - 114y_{n+2}, 721y_{n+2} - 4560x_{n+2})</td>
<td>Y^2 - 80X^2 = 2</td>
</tr>
<tr>
<td>11</td>
<td>(x_n, x_{n+1} - 19x_n)</td>
<td>Y^2 - 240X^2 = 6</td>
</tr>
<tr>
<td>12</td>
<td>(x_n, x_{n+2} - 721x_n)</td>
<td>Y^2 - 519840X^2 = 12996</td>
</tr>
<tr>
<td>13</td>
<td>(114x_{n+1} - 3x_{n+2}, 19x_{n+2} - 721x_{n+1})</td>
<td>Y^2 - 40X^2 = 9</td>
</tr>
</tbody>
</table>

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the table 2 below:

<table>
<thead>
<tr>
<th>S.no</th>
<th>(X,Y)</th>
<th>Parabola</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(y_{n+1} - y_n, 2y_{n+1})</td>
<td>X^2 = 90Y - 180</td>
</tr>
<tr>
<td>2</td>
<td>(y_{n+2} - 721y_n, 2y_{n+2})</td>
<td>X^2 = 129960Y - 259920</td>
</tr>
<tr>
<td>3</td>
<td>(x_{n+1} - 3y_n, 2y_{n+1})</td>
<td>160X^2 = 361Y - 722</td>
</tr>
<tr>
<td>4</td>
<td>(x_{n+2} - 114y_n, 2y_{n+2})</td>
<td>160X^2 = 519840Y - 1039682</td>
</tr>
<tr>
<td>5</td>
<td>(38y_{n+2} - 1442y_{n+1}, 114y_{n+2} - 3y_{n+4})</td>
<td>X^2 = 240Y - 720</td>
</tr>
<tr>
<td>6</td>
<td>(x_n, y_{n+2} - 120x_{n+1})</td>
<td>3040X^2 = 2Y - 38</td>
</tr>
<tr>
<td>7</td>
<td>(19x_{n+1} - 3y_{n+1}, 19y_{n+2} - 120x_{n+2})</td>
<td>80X^2 = Y - 1</td>
</tr>
<tr>
<td>8</td>
<td>(19x_{n+2} - 114y_{n+1}, 721y_{n+2} - 120x_{n+4})</td>
<td>80X^2 = 19Y - 361</td>
</tr>
</tbody>
</table>

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In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation:

\[ \frac{x^2}{m^2-n^2} - \frac{y^2}{m^2+n^2} = 1 \]

Consider \( m = x_{n+1} + y_{n+1}, n = x_{n+1} \). Observe that \( m > n > 0 \). Treat \( m, n \) as the generators of the Pythagorean triangle \( T(\alpha, \beta, \gamma) \), \( \alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2 \).

Then the following interesting relations are observed:

a) \( \alpha - 20\beta + 19\gamma = -1 \)

b) \( 2\alpha - \beta = 80A \)

c) \( 11\alpha - 10\beta + 9\gamma + 1 = 40A \)

d) \( \frac{2A}{P} = x_{n+1}y_{n+1} \)

CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation \( y^2 = 40x^2 + 1 \). As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive Pell equations and determine their integer solutions along with suitable properties.

REFERENCES

10. Gopalan MA, Yamuna RS. Remarkable observation on the binary quadratic equation \( y^2 = (k + 1)x^2 + 1, k \in \mathbb{Z} - \{0\} \), "Impact Journal of Science and Technolgy, 2010, Vol No.4,61-65.


15. Gopalan MA, Vidhyalakshmi S, Umarani J. Remarkable observation on the hyperbola \( y^2 = 24x^2 + 1 \),” Bulletin of mathematics and Statistics Research, 2014,


19. Goplan MA, Sivagami B. Special Pythagorean triangle generated through the integral solutions of the equation \( y^2 = (k^2 + 2x)x^2 + 1 \), Diaphantus J.Math., 2(1), 2013, 25-30.
