A Note on Existence of Positive Solutions of One-Dimensional \( p \)-Laplacian Equations

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Abstract: In this paper, we proved some new results on the existence of positive solutions of one-dimensional \( p \)-Laplacian equations under sublinear conditions. Our assumptions are weaker than the existing that, some recent results are improved essentially.

Keywords: positive solutions, one-dimensional \( p \)-Laplacian, sublinear conditions, negative values, fixed Point

INTRODUCTION

We investigate existence results of positive (classical) solutions for one-dimensional \( p \)-Laplacian equations of the form

\[
\begin{cases}
-\Delta_p z(x) = f(x, z(x)) \text{ for a.e. } x \in (0, 1) \\
z(0) = z(1) = 0
\end{cases}
\]  

(1.1)

under the sublinear conditions, where \( p \in (1, \infty) \), \(-\Delta_p z(x) = \left| z'(x) \right|^{p-2} z'(x) \right) \right|^{''} := \phi_p \left( z'(x) \right) \right) \right|^{''} \) \), \( z'(x) \) denotes the usual derivative of the function \( z \) at \( x \), and \( \phi_p : \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( \phi_p (s) = |s|^{p-2} s \).

Existence of nonzero nonnegative positive solution of (1.1) has been studied by many authors, for example, by Dai and Ma [1], where the nonlinearity is of the form \( g(x, f(u)) \), under the following condition:

\[
\lim_{u \to +\infty} \frac{f(x, u)}{\phi_p (u)} \geq 0 \quad \text{for } x \in (0, 1),
\]

and by Lan and Yang [2], where \( p \neq 2 \), under the following sublinear condition:

\[
\overline{f}(u) = \sup_{x \in [0, 1]} f(x, u), \quad \lim_{u \to +\infty} \sup \overline{f}(u) / \phi_p (u) \geq 0.
\]

When the first eigenvalue \( \mu_n \) with \( n = 1 \), Ćwiszewski and Maciejewski [3] use the Granas fixed point index to study the existence of positive weak solutions of \( p \)-Laplacian equations under the following sublinear condition:

\[
\lim_{u \to +\infty} f(x, u) / \phi_p (u) \in L^\infty.
\]

We refer to [1, 4, 5, 6] for the study of the existence and uniqueness of systems of \( p \)-Laplacian equations under superlinear or sublinear conditions. As mentioned above, there have been many papers studying the existence of solutions of one-dimensional \( p \)-Laplacian equation (1.1), but to the best of our knowledge, then essential condition \( f_p^\infty \) is required to be bounded below (numerical or functional lower bounds) such as Theorem 4.1 [1], Theorem 2.2 [2], Theorem 1.1 [3]. In this paper, we do not need this assumption, and our assumptions on (1.1) are weaker than usual that. The obtained results can be not derived by the existing results and some recent results are improved fundamentally. Regarding the study of (1.1), one may obtain more results referring to [1, 2, 3, 7] and their references.
New results of positive solutions for (1.1)
In this paper, we always assume the following conditions hold.
\((C_1)\) \(f : [0,1] \times R \rightarrow R\) is a Carathéodory function, that is, \(f (\cdot, u)\) is measurable for each fixed \(u \in R\), \(f (x, \cdot)\) is continuous for almost every (a.e.) \(x \in [0,1]\).
\((C_2)\) For each \(r > 0\), there exists \(g_r \in L^1 [0,1]\) such that \(|f (x, u)| \leq g_r (x)\) for a.e. \(x \in [0,1]\) and all \(u \in [0, r]\).
\((C_3)\) \(f (x, 0) \geq 0\) for a.e. \(x \in [0,1]\).

**Remark 2.1** The conditions \((C_1) - (C_3)\) have been used usually, for example in [2, 3]. The condition \((C_3)\) is weaker than usual assumptions, see Remark 2.1 [8, 9]. We denote by AC\([0,1]\) the space of all the absolutely continuous functions defined on \([0,1]\). A function \(z : [0,1] \rightarrow R\) is said to be a (classical) solution of (1.1) if \(z \in C[0,1], \phi_r (z') \in AC[0,1]\) and \(z\) satisfies (1.1) [7], to be nonnegative if \(z(x) \geq 0\) for \(x \in [0,1]\) and to be positive if \(z(x) > 0\) for \(x \in (0, 1)\).

We denote by \(W^{1,p} (0,1)\) the standard Sobolev space with norm \([10]\)
\[
\|u\|_{W^{1,p}} = \left( \int_{0}^{1} [u']^p (x) \, dx \right)^{1/p} := \|u\|_{W^{1,p}}.
\]
We need to recall some facts and prove several Lemmas. First, the following facts \(W^{1,p} (0,1) \subseteq C[0,1]\) and \(||u||_{W^{1,p}} \leq c_q ||u||_{L^p}\) for \(u \in W^{1,p}(0,1)\) \([11]\),Lemma A.9 (ii), Page 56 will be used to study (1.1), where \(c_q > 0\) is a constant.

**Lemma 2.1** ([12]) For every \(w \in L^1 (0,1)\), the quasilinear boundary value problem
\[
\begin{align*}
- \Delta u (x) &= w (x) \quad \text{for a.e. } x \in (0,1) \\
u (0) &= u (1) = 0
\end{align*}
\] has a unique solution \(u\) in \(D(\Delta_r)\), where
\(D(\Delta_r) = \{ u \in C^1 [0,1] : \phi_r (u') \in AC[0,1]\}\)
and \(C^1 [0,1] = \{ u \in C^1 [0,1] : u (0) = u (1) = 0 \}\) is a Banach space with the norm \(\|u\|_{C^1} = \|u\|_{[0,1]} + \|u'\|_{[0,1]}\).

We denote by \(T\) the inverse of \(-\Delta_r\). Then \(T : L^1 (0,1) \rightarrow D(\Delta_r)\) is defined by
\[
Tw = u
\]
for \(u\) is the unique solution of (2.2) in \(D(\Delta_r)\).

It is easy to see that the inverse operator \(T\) has the following property [2].
\[
T (Tw) = T^2 (w) \quad \text{for } w \in L^1 (0,1) \text{ and } t \geq 0.
\]

**Lemma 2.2** The map \(T : L^1 (0,1) \rightarrow D(\Delta_r)\) is increasing, that is, \(w_1, w_2 \in L^1 (0,1), w_1 \leq w_2\) implies \(Tw_1 \leq Tw_2\) and \(Tw \geq 0\) for \(w \in L^1 (0,1)\).

**Proof.** By Lemma 2.1, we may assume that \(u_i \in D(\Delta_i), i = 1, 2\) such that \(Tw_i = u_i, i = 1, 2\). Then
\[
- \Delta u_i (x) = w_i (x), \quad i = 1, 2
\]
Since \(u_i (0) = u_i (1)\), we have \(\xi \in (0,1)\) such that \(u_i' (\xi) = u_i' (\xi)\). By (2.5), we have
\[
\begin{align*}
u_i (x) &= \int_{0}^{\xi} \phi_r \left[ \int_{0}^{\xi} w_i (\tau) \, d\tau + \phi_r (u_i' (\xi)) \right] \, dx, \quad 0 \leq x \leq \xi \\
&= \int_{0}^{\xi} \phi_r \left[ \int_{0}^{\xi} w_i (\tau) \, d\tau - \phi_r (u_i' (\xi)) \right] \, dx, \quad \xi < x \leq 1
\end{align*}
\]
Since $\phi^+_g$, the inverse function of $\phi_g$ is an increasing function, then $w_i \leq w_2$ implies that $u_i \leq u_2$ and $Tw_i \leq Tw_2$. Let $w_i = 0$ and $-\Delta u = w \in L^1(0,1)$. Then $u \geq u_0 = 0$ and $Tw = u \geq 0$.

**Lemma 2.3** ([13]) For each $g \in L^1(0,1)$ with \( \int_0^1 g(x) dx > 0 \), there exists $u_0 > 0$ and $\phi_g \in C^1_0[0,1] \cap (P \setminus \{0\})$ satisfying

\[
\begin{align*}
-\Delta \phi_g(x) &= \mu_g g(x) \phi_g(x) \quad \text{for a.e. } x \in (0,1), \\
\phi_g(0) &= \phi_g(1) = 0.
\end{align*}
\]  

(2.7)

The positive value $\mu_g$ is called the first eigenvalue of (2.7), $\phi_g$ is called the eigenfunction corresponding to the eigenvalue $\mu_g$ and we see that for $g \in L^1(0,1) \setminus \{0\}$,

\[
\mu_g = \inf \left\{ \frac{\int_0^1 \|u'(x)\|^2 + \int_0^1 g(x)\|u(x)\|^2}{\int_0^1 \|u(x)\|^2} : u \in W^{1,2}_{0,1}(0,1) \right\},
\]

(2.8)

where $\int_0^1 \|u'(x)\|^2 + \int_0^1 g(x)\|u(x)\|^2 dx = \infty$ if $\int_0^1 g(x)\|u(x)\|^2 dx = 0$. It is given in [7] that the first eigenvalue $\mu_g$ with $g \equiv 1$ equals

\[
\mu_g(1) = 2\pi^2.
\]

(2.9)

Define a map $A$ from $W^{1,2}_{0,1}$ to $D(\Lambda_1)$ by

\[
A\zeta(x) = (TF\zeta)(x)
\]

(2.10)

where $T$ is given in (2.3) and the Nemytskii operator $F : C[0,1] \to L^1(0,1)$ is defined by

\[
F\zeta(x) = f(x, r\zeta(x)),
\]

(2.11)

where $r\zeta(x) = \max\{\zeta(x), 0\}$.

**Lemma 2.4** Assume that $\zeta \neq 0$ and is (classical) solution of following problem

\[
\begin{align*}
-\Delta \zeta(x) &= \nu f(x, r\zeta(x)) \quad \text{for a.e. } x \in (0,1) \\
\zeta(0) &= \zeta(1) = 0
\end{align*}
\]

(2.12)

where $\nu > 0$. Then the following assertions hold.

(i) \( \zeta(x) \geq 0 \text{ for } x \in (0,1) \)

(ii) \( \text{If there exists } \rho > 0 \text{ such that } f(x, u) \geq 0 \text{ for a.e. } (x, u) \in [0,1] \times [0, \rho] \).

Then $\zeta(x) > 0$ for $x \in (0,1)$.

**Proof.** (i) If there exists $x_0 \in (0,1)$ such that $\zeta(x_0) < 0$, by $\zeta(0) = \zeta(1) = 0$, we may choose $[a, b] \subseteq [0,1]$ such that $x_0 \in (a, b)$, $\zeta(x) < 0$ on $(a, b)$ and $\zeta(a) = \zeta(b) = 0$.

Let $\xi \in (a, b)$ such that $\zeta(\xi) = \min \{\zeta(x) : x \in [a, b]\} < 0$, then $\zeta'(\xi) = 0$.

Integrating (2.12) from $x$ to $\xi$ and noticing $r\zeta(x) = 0$ on $[a, b]$, we have

\[
\Phi_\nu(z'(x)) = \int_0^x f(t, r\zeta(t)) dt = \int_0^x f(t, 0) dt \geq 0 \text{ for } x \in [a, \xi]
\]

and

\[
\Phi_\nu(z'(x)) = \int_0^x f(t, 0) dt \geq 0 \text{ for } x \in [a, \xi].
\]

Then, we obtain $z'(x) \geq \Phi_\nu(0) = 0$ on $[a, \xi]$, that is, $\zeta(x)$ is increasing on $[a, \xi]$ and $\zeta(\xi) \geq \zeta(a)$, a contradiction. Hence $\zeta(x) \geq 0$ on $[0,1]$.

(ii) If there exists $x \in (0,1)$ such that $\zeta(x) = 0$, without loss of generality, we may assume by $\zeta(x) \neq 0$ and $\zeta(x) \geq 0$ on $[0,1]$ that there exists $a \in (0, x_0)$ such that $\zeta(a) > 0$, $\zeta(x_0) = 0$ and $\zeta(x) \leq \rho$ on $[a, x_0]$.
Integrating (2.12) from $x$ to $x_0$ and noticing $z'(x_0) = 0$. We obtain easily
\[
\Phi_r(z(x)) = \int_0^x f(t,0) \, dt \geq 0 \quad \text{for} \quad x \in [a, x_0].
\]
From this we have $z'(x) \geq \Phi_r^{-1}(0) = 0$ on $[a, x_0]$ and $z(x_0) \geq z(a) > 0$, it is a contradiction. Hence $z(x) > 0$ for $x \in (0,1)$. □

Lemma 2.5 Under $(C_1) - (C_2)$, the following assertions hold.

(i) The map $A$ defined in (2.10) maps $\mathcal{W}_0^{1,p}(0,1)$ into $\mathcal{W}_0^{1,p}(0,1)$ and is compact.

(ii) $z \in \mathcal{W}_0^{1,p}(0,1)$ is a fixed point of $A$ if and only if $z$ is a nonnegative solution of (1.1).

Proof. (i) The proof is very similar to that of Theorem 2.1 (i) in [2] and is omitted.

(ii) If $z \in \mathcal{W}_0^{1,p}(0,1)$ is a fixed point of $A$, then $z(x) \geq 0$ for $0 \leq x \leq 1$ by Lemma 2.4 (i). Since $T$ maps $L^1(0,1)$ into $C[0,1]$, and thus, by (2.3), $z \in C[0,1]$ and is a nonnegative solution of (1.1).

The inverse is true obviously.

We shall use the following known result (for example [14], which can be proved by using Leray-Schauder degree theory for compact maps in Banach spaces).

Lemma 2.6 Let $E$ be a Banach space, $\Omega_1$ and $\Omega_2$ be two bounded open sets of $E$, $\theta \in \Omega_1 \subset \Omega_2$, where $\theta$ is zero element of $E$. Assume that $A: \Omega_1 \setminus \Omega_2 \rightarrow E$ is compact and satisfies

(1) There exists $\gamma_0 \in E \setminus \{0\}$ such that $x \neq Ax + t\gamma_0$ for $x \in \partial \Omega_1$ and $0 < t \leq 1$.

Then $A$ has a fixed point in $\Omega_1 \setminus \overline{\Omega_2}$.

Now, we state and prove new results of positive solutions for (1.1).

Theorem 2.1 Assume that $(C_1) - (C_3)$ and the following conditions hold.

(H$_1$) There exists $\rho_0 > 0$, $\varepsilon_0 > 0$ and $\gamma_0 \in L^1(0,1) \setminus \{0\}$ such that
\[
f(x,u) \geq \left( \mu_{\gamma_0} + \varepsilon_0 \right) \, \gamma_0(x) u^{\rho_0-1} \quad \text{for a.e.} \quad x \in [0,1]
\]
and all $u \in [0, \rho_0]$.

(H$_2$) There exist $\mu_0 > 0$, $\varepsilon_2 \in (0, \mu_0)$ and $\phi_0 \in L^1(0,1) \setminus \{0\}$ such that
\[
f(x,u) \leq \left( \mu_0 - \varepsilon_2 \right) \, \phi_0(x) u^{\rho_0-1} \quad \text{for a.e.} \quad x \in [0,1]
\]
and all $u \in [\mu_0, \infty]$.

Then (1.1) has a positive solution $z$ in $C_0^1[0,1]$, that is, $z \in C_0^1[0,1]$ satisfies $z(x) > 0$ for $x \in (0,1)$.

Proof. Let $\rho = \rho_0$ and $\Omega_\rho = \{ z : z \in \mathcal{W}_0^{1,p}(0,1), \|z\| \leq \rho \}$. If there exists $z \in \partial \Omega_\rho$ such that $z = T(Fz)$, then Lemma 2.4 (ii) (see H$_2$) implies that the results of Theorem 2.1 holds. Hence, we assume that $z \neq T(Fz)$ for $z \in \partial \Omega_\rho$ and prove that
\[
z \neq T(Fz) + te \quad \text{for} \quad z \in \partial \Omega_\rho \quad \text{and} \quad t > 0,
\]
where $e$ is the eigenfunction corresponding to the eigenvalue $\mu_{\gamma_0}$, that is,
\[
\begin{aligned}
-\Delta e(x) &= \mu_{\gamma_0} \gamma_0(x) e^{\rho_0-1}(x) \quad \text{for a.e.} \quad x \in (0,1), \\
e(0) &= e(1) = 0.
\end{aligned}
\]
In fact, if not, there exists $z \in \partial \Omega_\rho$ and $t > 0$ such that $z = T(Fz) + te$. By (2.1), we have for $z \in \partial \Omega_\rho$,
\[
\|z\| \leq \|e\|_{L^\infty} \leq c_0 \|\gamma_0\|_{L^\infty} = c_0 \rho = \rho_0 \quad \text{for} \quad x \in [0,1]
\]
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Hence $0 \leq rz(x) \leq \rho_n$, by $(H_r)$ and, and $F_z(x) = F(x, rz(x)) \geq 0$ for $x \in [0,1]$. From this and Lemma 2.2 we know $z \geq \tau e$. Let $\tau = \sup \{ v : z \geq ve \}$. Then $0 < \tau < \infty$ and $z \geq \tau e$.

It follows from $(H_r)$ and (2.14) that for a.e. $x \in [0,1]$

$$f(x, z(x)) \geq (\mu_{v_n} + \epsilon) \psi_{v_n}(x) z^{-1}(x) \geq (\mu_{v_n} + \epsilon) \tau^{-1} \psi_{v_n}(x) e^{-1}(x).$$

(2.15)

By Lemma 2.2, (2.4) and (2.15), we have

$$z(x) \geq TF_z(x) \geq T \left[ (\mu_{v_n} + \epsilon) \tau^{-1} \psi_{v_n}(x) e^{-1}(x) \right] = \tau \left( (\mu_{v_n} + \epsilon) \tau^{-1} T \psi_{v_n}(x) e^{-1}(x) \right).$$

Noticing that $\mu_{v_n}^{-1} e(x) = T \left[ \psi_{v_n}(x) e^{-1}(x) \right]$, we have

$$z(x) \geq \tau \left( (\mu_{v_n} + \epsilon) \tau^{-1} \mu_{v_n}^{-1} e(x) \right).$$

The definition of $\tau$ implies $\tau \geq \tau \left( (\mu_{v_n} + \epsilon) \tau^{-1} \mu_{v_n}^{-1} > \tau \right.$, it is a contradiction. Hence (2.13) holds.

By $(C_z)$, there exists $g_\mu \in L^1[0,1]$ such that $|f(x,u)| \leq g_\mu(x)$ for a.e. $x \in [0,1]$ and all $u \in [0,\epsilon]$.

This together with $(H_\mu)$, implies, for a.e. $x \in [0,1]$ and all $u \in R$

$$f(x,u) \leq g_\mu(x) + \left( \mu_{v_n} - \epsilon_2 \right) \phi_\mu(x) u^{-1}. \quad \text{(2.16)}$$

Let $r = \left( \epsilon_1 \epsilon_2 \mu_{v_n} \right) \| g_\mu \|_L^{1/\theta} / r > r$ and $\Omega = \{ z : z \in W^1_{q^*}(0,1), \| z \|_r < \rho \}$. Clearly, $r > c_{n_0} \rho_0 = \rho$. We prove that

$$z \not\in t AZ \text{ for } z \in \partial \Omega \text{ and } t \in [0,1]$$

(2.17)

If (2.17) is false, then there exists $z \in \partial \Omega$ and $t \in [0,1]$ such that $z = t AZ$. It follows from (2.4) and (2.3) that $z(x) = T(tAZ) (x)$ and $-\Delta_z z(x) = tAZ \phi_z(x)$ for a.e. $x \in [0,1]$. By Lemma 2.4 (i), we have $z(x) \geq 0$ for $x \in (0,1)$. Hence

$$-\Delta_z z(x) = tAZ \phi_z(x) \text{ for a.e. } x \in [0,1].$$

(2.18)

By (2.18), (2.16), (2.8) with $g = \phi_\mu$ and (2.1), we have

$$\| x \|_{w^{1,q}} = \left( -\Delta_z z \right) = tAZ \phi_z (x) \| x \|_{w^{1,q}} \leq \left( g_\mu(x) + \left( \mu_{v_n} - \epsilon_2 \right) \phi_\mu(x) z^{-1}(x) \right) \| x \|_{w^{1,q}}$$

$$\leq \| g_\mu \|_L^{1/\theta} \| g_\mu \|_L^{1/\theta} \left[ g_\mu(x) + \left( \mu_{v_n} - \epsilon_2 \right) \mu_{v_n}^{1/\theta} \right] \| x \|_{w^{1,q}}$$

$$\leq c_{n_0} \| g_\mu \|_L^{1/\theta} \| g_\mu \|_L^{1/\theta} \left[ g_\mu(x) + \left( 1 - \epsilon_2 \mu_{v_n}^{1/\theta} \right) \right] \| x \|_{w^{1,q}}$$

This implies that $\| x \|_{w^{1,q}} \leq \epsilon_1 \epsilon_2 \mu_{v_n} \| g_\mu \|_L^{1/\theta}$. Hence, we have

$$r_1 < r \leq \left( \epsilon_1 \epsilon_2 \mu_{v_n} \| g_\mu \|_L^{1/\theta} \right)^{-1} = r_1,$$

a contradiction. Hence (2.17) holds.

By Lemma 2.5 (i) and Lemma 2.6, there exists $z \in \Omega \setminus \overline{\Omega}_p$ such that $z = AZ$ and by Lemma 2.5 (ii), $z$ is a nonnegative solution of (1.1). It follows from $(H_r)$ and Lemma 2.4 (ii) that $z(x) > 0$ for $x \in (0,1)$, that is, $z$ is a positive solution of (1.1).

Let $E$ be a fixed subset of $[0,1]$ of measure zero and
\[ \overline{f}(u) = \sup_{x \in [0,1]} f(x, u), \quad \underline{f}(u) = \inf_{x \in [0,1]} f(x, u). \]

Notation:

\[ (f_\alpha)_\alpha = \lim_{\alpha \to \infty} \frac{f(u)}{\alpha}, \quad (f^*_\alpha)_\alpha = \lim_{\alpha \to \infty} \overline{f}(u), \quad (f^*_{\alpha})_\alpha = \lim_{\alpha \to \infty} \inf_{x \in [0,1]} f(x, u). \]

Corollary 2.1 Assume that \((C_1), (C_3)\) and the following conditions hold.

\[ f_\alpha < \mu_\alpha(p) < (f_\alpha)_\alpha \leq \infty, \quad (2.19) \]

where \(\mu_\alpha(p)\) is the same as in \((2.9)\).

Then \((1.1)\) has a positive solution \(z\) in \(C_0([0,1])\).

**Proof.** By \((2.19)\), \((H_1)\) with \(\phi_\alpha = 1\) and \((H_2)\) with \(\psi_\alpha = 1\) hold for some \(\alpha > 0(i = 1, 2)\), \(\rho_\alpha > 0\) and \(r_\alpha > 0\). The result follows from Theorem 2.1.

Example 2.1 Let \(f(u) = \lambda u^p (1 - p)(\lambda > \mu_\alpha(p))\), where \(\mu_\alpha(p)\) is the same as in \((2.9)\). Then \((1.1)\) has a positive solution in \(C_0([0,1])\).

**Proof.** \(f_\alpha = \lambda\) and \(f^*_{\alpha} = -\infty\), the result follows from Corollary 2.1.

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