The Eigenvalues and the Eigenfunctions of the Sturm-Liouville Fuzzy Boundary Value Problem According To the Generalized Differentiability
Hülya Gültekin Çitil
Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun, Turkey

Abstract: In this paper, the eigenvalues and the eigenfunctions of the fuzzy Sturm-Liouville fuzzy boundary value problem equation is examined under the approach of generalized differentiability. Different examples are solved for these problems.

Keywords: Fuzzy boundary value problems, generalized differentiability, eigenvalue, eigenfunction

1. Introduction
Fuzzy differential equations are studied by many researchers. Fuzzy differential equations and fuzzy boundary value problems are one of the major applications of fuzzy number arithmetic. The first approach is the use of Hukuhara differentiability to solve fuzzy differential equations. This approach has a drawback: the solution becomes fuzzier as time goes by [2, 5]. The second approach is the generalized differentiability. The generalized differentiability was introduced [1] and studied in [2-4, 7, 10].

Gültekin Çitil and Albıyıçık [6] have defined the fuzzy Sturm-Liouville equation and they have examined eigenvalues and eigenfunctions of the fuzzy Sturm-Liouville problem under the approach of the Hukuhara differentiability.

In this paper, a investigation is made on the eigenvalues and the eigenfunctions of the fuzzy Sturm-Liouville problem by using generalized differentiability.

2. Preliminaries
Definition 2.1. [7] A fuzzy number is a function u: \( \mathbb{R} \to [0,1] \) satisfying the following properties:
- u is normal, convex fuzzy set, upper semi-continuous on \( \mathbb{R} \) and \( \text{cl}\{x \in \mathbb{R} | u(x) > 0\} \) is compact, where \( \text{cl} \) denotes the closure of a subset.

Let \( \mathbb{R}_f \) denote the space of fuzzy numbers.

Definition 2.2. [8] Let \( u \in \mathbb{R}_f \). The \( \alpha \)-level set of u, denoted \( [u]^\alpha \), \( 0 < \alpha \leq 1 \), is \( [u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\} \). If \( \alpha = 0 \), the support of u is defined \( [u]^0 = \text{cl}\{x \in \mathbb{R} | u(x) > 0\} \). The notation, \( [u]^\alpha = [u^-_\alpha, u^+_\alpha] \) denotes explicitly the \( \alpha \)-level set of u. We refer to \( u^- \) and \( u^+ \) as the lower and upper branches of u, respectively.

The following remark shows when \( [u^-_\alpha, u^+_\alpha] \) is a valid \( \alpha \)-level set.

Remark 2.1. [7] The sufficient and necessary conditions for \( [u^-_\alpha, u^+_\alpha] \) to define the parametric form of a fuzzy number as follows:

1) \( u^-_\alpha \) is bounded monotonic increasing (nondecreasing) left-continuous function on \( (0,1] \) and right-continuous for \( \alpha = 0 \),

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2) \( \mu_{u,\alpha} \) is bounded monotonic decreasing (nonincreasing) left-continuous function on \((0,1]\) and right-continuous for \( \alpha = 0 \).

3) \( \mu_{u,\alpha} \leq \mu_{u,\alpha} , 0 \leq \alpha \leq 1 \).

**Definition 2.3.** [9] If \( A \) is a symmetric triangular number with support \([a, a] \), the \( \alpha \)-level set of \( A \) is

\[
[A]^{\alpha} = \left[ a + \frac{a-a}{2} \alpha , a - \frac{a-a}{2} \alpha \right].
\]

**Definition 2.4.** [8] For \( u, v \in \mathbb{R}_F \) and \( \lambda \in \mathbb{R} \), the sum \( u + v \) and the product \( \lambda u \) are defined by \([u + v]^{\lambda} = [u]^{\lambda} + [v]^{\lambda}\), \([\lambda u]^{\alpha} = \lambda [u]^{\alpha} , \forall \alpha \in (0,1)\), where \([u]^{\alpha} + [v]^{\alpha}\) means the usual addition of two intervals (subsets) of \( \mathbb{R} \) and \([\lambda u]^{\alpha}\) means the usual product between a scalar and a subset of \( \mathbb{R} \).

The metric structure is given by the Hausdorff distance

\[
D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R} \cup \{0\},
\]

by

\[
D(u, v) = \sup_{\alpha \in (0,1)} \max \{ |\mu_{a,\alpha} - \nu_{a,\alpha}| , |\mu_{u,\alpha} - \nu_{u,\alpha}| \}. [7].
\]

**Definition 2.5.** [9] Let \( u, v \in \mathbb{R}_F \). If there exist \( w \in \mathbb{R}_F \) such that \( u = v + w \), then \( w \) is called the H-difference of \( u \) and \( v \) and it is denoted \( u - v \).

**Definition 2.6** [8] Let \( f: [a, b] \to \mathbb{R}_F \) and \( t_0 \in [a, b] \). We say that \( f \) is (1)-differentiable at \( t_0 \), if there exists an element \( f'(t_0) \in \mathbb{R}_F \) such that for all \( h > 0 \) sufficiently small near to 0, exist \( f(t_0 + h) - f(t_0) , f(t_0) - f(t_0 - h) \) and the limits

\[
\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) - f(t_0 - h)}{-h} = f'(t_0),
\]

and \( f \) is (2)-differentiable if for all \( h > 0 \) sufficiently small near to 0, exist \( f(t_0) - f(t_0 + h) , f(t_0 - h) - f(t_0) \) and the limits

\[
\lim_{h \to 0} \frac{f(t_0) - f(t_0 + h)}{-h} = \lim_{h \to 0} \frac{f(t_0 - h) - f(t_0)}{h} = f'(t_0).
\]

**Theorem 2.1.** [7] Let \( f: [a, b] \to \mathbb{R}_F \) be fuzzy function, where \([f(t)]^{\alpha} = [f_{-\alpha}(t), f_{+\alpha}(t)]\), for each \( \alpha \in [0,1] \).

(i) If \( f \) is (1)-differentiable then \( f_{-\alpha} \) and \( f_{+\alpha} \) are differentiable functions and \([f'(t)]^{\alpha} = [f'_{-\alpha}(t), f'_{+\alpha}(t)]\).

(ii) If \( f \) is (2)-differentiable then \( f_{-\alpha} \) and \( f_{+\alpha} \) are differentiable functions and \([f'(t)]^{\alpha} = [f'_{-\alpha}(t), f'_{+\alpha}(t)]\).

**Theorem 2.2.** [7] Let \( f': [a, b] \to \mathbb{R}_F \) be fuzzy function, where \([f(t)]^{\alpha} = [f_{-\alpha}(t), f_{+\alpha}(t)]\), for each \( \alpha \in [0,1] \). \( f \) is (1)-differentiable or (2)-differentiable.

(i) If \( f \) and \( f' \) are (1)-differentiable then \( f'_{-\alpha} \) and \( f'_{+\alpha} \) are differentiable functions and

\[
[f'(t)]^{\alpha} = [f'_{-\alpha}(t), f'_{+\alpha}(t)].
\]
If $f$ is (1)-differentiable and $f'$ is (2)-differentiable then $\overline{f_a}$ and $\overline{f_a}'$ are differentiable functions and

$$[f^{-}(t)]^n = \left[ \overline{f_a(t)}, \overline{f_a}'(t) \right],$$

If $f$ is (2)-differentiable and $f'$ is (1)-differentiable then $\overline{f_a}$ and $\overline{f_a}'$ are differentiable functions and

$$[f^{-}(t)]^n = \left[ \overline{f_a(t)}, \overline{f_a}'(t) \right],$$

If $f$ and $f'$ are (2)-differentiable then $\overline{f_a}$ and $\overline{f_a}'$ are differentiable functions and

$$[f^{-}(t)]^n = \left[ \overline{f_a(t)}, \overline{f_a}'(t) \right].$$

3. The Eigenvalues and The Eigenfunctions of The Sturm-Liouville Fuzzy Boundary Value Problem According to The Generalized Differentiability

Consider the eigenvalues and the eigenfunctions of the fuzzy boundary value problem

$$L y = p(x) y'' + q(x) y$$

$$L y + \lambda y = 0, \quad x \in (a, b),$$

$$B_1(y) := Ay(a) + By'(a) = 0,$$

$$B_2(y) := Cy(b) + Dy'(b) = 0$$

using the generalized differentiability, where $p(x), q(x)$ are continuous functions and are positive on $[a, b]$, $p'(x) = 0$, $\lambda > 0$, $A, B, C, D \geq 0$, $A^2 + B^2 \neq 0$ and $C^2 + D^2 \neq 0$. Here, (i,j) solution means that $y$ is (i) differentiable and $y'$ is (j) differentiable, $i,j=1,2$.

For (1,1) solution,

$$p(x) \left[ y''_a(x), \overline{y''_a}(x) \right] + q(x) \left[ y''_a(x), \overline{y''_a}(x) \right] + \lambda \left[ y''_a(x), \overline{y''_a}(x) \right] = 0,$$

$$A \left[ y_a(a), \overline{y_a}(a) \right] + B \left[ y'_a(a), \overline{y'_a}(a) \right] = 0,$$

$$C \left[ y_a(b), \overline{y_a}(b) \right] + D \left[ y'_a(b), \overline{y'_a}(b) \right] = 0,$$

for (1,2) solution,

$$p(x) \left[ y''_a(x), \overline{y''_a}(x) \right] + q(x) \left[ y''_a(x), \overline{y''_a}(x) \right] + \lambda \left[ y''_a(x), \overline{y''_a}(x) \right] = 0,$$

$$A \left[ y_a(a), \overline{y_a}(a) \right] + B \left[ y'_a(a), \overline{y'_a}(a) \right] = 0,$$

$$C \left[ y_a(b), \overline{y_a}(b) \right] + D \left[ y'_a(b), \overline{y'_a}(b) \right] = 0,$$

for (2,2) solution;

$$p(x) \left[ y''_a(x), \overline{y''_a}(x) \right] + q(x) \left[ y''_a(x), \overline{y''_a}(x) \right] + \lambda \left[ y''_a(x), \overline{y''_a}(x) \right] = 0,$$

$$A \left[ y_a(a), \overline{y_a}(a) \right] + B \left[ y'_a(a), \overline{y'_a}(a) \right] = 0,$$

$$C \left[ y_a(b), \overline{y_a}(b) \right] + D \left[ y'_a(b), \overline{y'_a}(b) \right] = 0,$$

for (2,1) solution;

$$p(x) \left[ \overline{y''_a}(x), \overline{y''_a}(x) \right] + q(x) \left[ \overline{y''_a}(x), \overline{y''_a}(x) \right] + \lambda \left[ \overline{y''_a}(x), \overline{y''_a}(x) \right] = 0,$$
\[ A \left[ y_a(a), y'_a(a) \right] + B \left[ y'_a(a), y''(a) \right] = 0, \]
\[ C \left[ y_a(b), y'_a(b) \right] + D \left[ y'_a(b), y''(b) \right] = 0, \]
are solved.

**Example 3.1.** Consider the fuzzy Sturm-Liouville problem
\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0. \]  
(3.4)

Let be \( \lambda = k^2, \ k > 0 \). For (1.1) and (2.2) solution,
\[ \left[ y''_a, y'_a \right] + \lambda \left[ y'_a, y_a \right] = 0 \]
\[ y_a(0), y'_a(0) = 0, \quad y_a(1), y'_a(1) = 0 \]
is solved. Then, the solution of fuzzy differential equation is
\[ y(x) = c_1(\alpha) \cos(kx) + c_2(\alpha) \sin(kx), \]  
(3.5)
\[ \left[ y(x) \right]^{\alpha} = \left[ y_a(x), y'_a(x) \right], \]  
(3.6)
\[ \frac{\partial(c_2(\alpha) \sin(n\pi x))}{\partial \alpha} \geq 0, \quad \frac{\partial(c_4(\alpha) \sin(n\pi x))}{\partial \alpha} \leq 0 \]
\[ \left[ y(x) \right]^{\alpha} = \text{a valid } \alpha \text{- level set}. \]

**i)** If \( n \) is single, \( \sin \left( n \pi x \right) \geq 0 \). Then, for \( c_2(\alpha), \ c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \geq 0, \quad \frac{\partial(c_4(\alpha))}{\partial \alpha} \leq 0 \) and \( c_2(\alpha) \leq c_4(\alpha) \), \( \left[ y(x) \right]^{\alpha} \) is a valid \( \alpha \)-level set.

**ii)** If \( n \) is double, \( \sin \left( n \pi x \right) \leq 0 \). Then, for \( c_2(\alpha), \ c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \leq 0, \quad \frac{\partial(c_4(\alpha))}{\partial \alpha} \geq 0 \) and \( c_2(\alpha) \geq c_4(\alpha) \), \( \left[ y(x) \right]^{\alpha} \) is a valid \( \alpha \)-level set.

Consequently; \( n \pi x \in \left[ - \pi, \pi \right], \ n = 1,2,... \)

**i)** If \( n \) is single, for \( c_2(\alpha), \ c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \geq 0, \quad \frac{\partial(c_4(\alpha))}{\partial \alpha} \leq 0 \) and \( c_2(\alpha) \leq c_4(\alpha) \), the eigenvalues are \( \lambda_n = \frac{n \pi^2}{4}, \) with associated eigenfunctions \( y_n(x) = [c_2(\alpha) \sin(n\pi x), c_4(\alpha) \sin(n\pi x)]. \)

**ii)** If \( n \) is double, for \( c_2(\alpha), \ c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \leq 0, \quad \frac{\partial(c_4(\alpha))}{\partial \alpha} \geq 0 \) and \( c_2(\alpha) \geq c_4(\alpha) \), the eigenvalues are \( \lambda_n = \frac{n \pi^2}{4}, \) with associated eigenfunctions \( y_n(x) = [c_2(\alpha) \sin(n\pi x), c_4(\alpha) \sin(n\pi x)]. \)

**iii)** If \( \alpha = 1 \), the eigenvalues are \( \lambda_n = \frac{n \pi^2}{4}, \) with associated eigenfunctions
\[ \left[ y_n \left( x \right) \right]^{\alpha} = \sin \left( n \pi x \right). \]

For (1.2) and (2.1) solution,
\[ \left[ y''_a, y'_a \right] + \lambda \left[ y'_a, y_a \right] = 0 \]
\[ y_a(0), y'_a(0) = 0, \quad y_a(1), y'_a(1) = 0 \]
is solved. Then, the solution of fuzzy differential equation is
\[ y_a(x) = -a_1(\alpha) e^{kx} - a_2(\alpha) e^{-kx} + a_3(\alpha) \sin(kx) + a_4(\alpha) \cos(kx), \]  
(3.8)
\[ \overline{y}_a(x) = a_1(\alpha) e^{kx} + a_2(\alpha) e^{-kx} + a_3(\alpha) \sin(kx) + a_4(\alpha) \cos(kx), \]  
(3.9)
\[ [y(x)]^\alpha = \left[ y'_a(x), y''_a(x) \right]. \tag{3.10} \]

From the boundary conditions \( a_1(\alpha) = a_2(\alpha) = a_4(\alpha) = 0, \ a_3(\alpha) s(n(k) = 0 \) are obtained. From here,
\[ a_3(\alpha) \neq 0, \sin(k) = 0 \Rightarrow k_n = n\pi, n = 1,2, \ldots \]
The eigenvalues are \( \lambda_n = n^2 \pi^2 \), with associated eigenfunctions
\[ [y_n(x)]^\alpha = \sin(n\pi x). \]

**Example 3.2.** If we take
\[ y(0) = 0, \ y'(1) = 0 \tag{3.11} \]
as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,
\[ \left[ y''_a, y''_a \right] + \lambda \left[ y'_a, y'_a \right] = 0 \]
\[ y'_a(0), y'_a(1) = 0 \]
is solved. Then, the solution of fuzzy differential equation is (3.5)-(3.7). From the first boundary condition \( c_1(\alpha) = c_3(\alpha) = 0, \) from the second boundary condition,
\[ k c_2(\alpha) \cos(k) = 0, k c_4(\alpha) \cos(k) = 0 \]
are obtained.

Then, \( c_2(\alpha) \neq 0, c_4(\alpha) \neq 0, \cos(k) = 0 \Rightarrow k_n = \frac{(2n-1)}{2}\pi, n = 1,2, \ldots \]
\[ [y(x)]^\alpha = \left[ c_2(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right), c_4(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right) \right]. \]

As
\[ \frac{\partial}{\partial \alpha} \left( c_2(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right) \right) \geq 0, \quad \frac{\partial}{\partial \alpha} \left( c_4(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right) \right) \leq 0 \]
and
\[ c_2(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right) \leq c_4(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right) \]
\[ [y(x)]^\alpha \] is a valid \( \alpha \)-level set. Let be \( \frac{(2n-1)}{2}\pi x \in \left[ \left( n - 1 \right) \pi, n\pi \right], n = 1,2, \ldots \)

i) If \( n \) is single, \( \sin \left( \frac{2n-1}{2}\pi x \right) \geq 0 \). Then, for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \geq 0, \ \frac{\partial(c_4(\alpha))}{\partial \alpha} \leq 0 \)
and \( c_2(\alpha) \leq c_4(\alpha), [y(x)]^\alpha \) is a valid \( \alpha \)-level set.

ii) If \( n \) is double, \( \sin \left( \frac{2n-1}{2}\pi x \right) \leq 0 \). Then, for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \leq 0, \ \frac{\partial(c_4(\alpha))}{\partial \alpha} \geq 0 \)
and \( c_2(\alpha) \geq c_4(\alpha), [y(x)]^\alpha \) is a valid \( \alpha \)-level set.

Consequently:
\[ \frac{2n-1}{2}\pi x \in \left[ \left(n - 1\right)\pi, n\pi \right], n = 1,2, \ldots \]

i) If \( n \) is single, for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \geq 0, \)
\[ \frac{\partial(c_4(\alpha))}{\partial \alpha} \leq 0 \] and \( c_2(\alpha) \leq c_4(\alpha), [y(x)]^\alpha \) the eigenvalues are \( \lambda_n = \frac{(2n-1)^2}{4}\pi^2 \), with associated eigenfunctions
\[ [y(x)]^\alpha = \left[ c_2(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right), c_4(\alpha) \sin \left( \frac{(2n-1)}{2}\pi x \right) \right]. \]

ii) If \( n \) is double, for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \leq 0, \)
If \( a_{1}(\alpha) = a_{2}(\alpha) = a_{4}(\alpha) = 0 \), \( a_{3}(\alpha) \cos(k) = 0 \),

\[ a_{3}(\alpha) \neq 0, \cos(k) = 0 \Rightarrow k_{n} = \frac{(2n-1)}{2} \pi, n = 1,2, \ldots \]

are obtained. The eigenvalues are \( \lambda_{n} = \frac{(2n-1)}{4} \pi^{2} \), with associated eigenfunctions

\[ \left[ y_{n}(x) \right]^{\alpha} = \sin \left( \frac{(2n-1)}{2} \pi x \right) . \]

For (2,1) solution,

\[ \left[ y_{a}, y_{b} \right] + \lambda \left[ y_{a}, y_{b} \right] = 0 \]

is solved. Eigenvalues and eigenfunctions are same with (1,2) solution.

Example 3.3. If we take

\[ y'(0) = 0, \quad y(1) = 0 \quad (3.12) \]

as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

\[ \left[ y_{a}^{\prime \prime}, y_{a}^{\prime} \right] + \lambda \left[ y_{a}, y_{a} \right] = 0 \]

\[ y_{a}(0), y_{a}(1) = 0, \quad y_{a}^{\prime}(1) = y_{a}^{\prime}(1) = 0 \]

is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the first boundary condition \( c_{2}(\alpha) = c_{4}(\alpha) = 0 \), from the second boundary condition

\[ y_{a}(1) = c_{1}(\alpha) \cos(k) = 0, \quad y_{a}(1) = c_{3}(\alpha) \cos(k) = 0 \]

are obtained.

Then, \( c_{1}(\alpha) \neq 0, c_{3}(\alpha) = 0 \), \( \cos(k) = 0 \Rightarrow k_{n} = \frac{(2n-1)}{2} \pi, n = 1,2, \ldots \)

\[ [y(x)]^{\alpha} = \left[ c_{1}(\alpha) \cos \left( \frac{(2n-1)}{2} \pi x \right), c_{3}(\alpha) \cos \left( \frac{(2n-1)}{2} \pi x \right) \right] . \]

\[ \frac{\partial}{\partial \alpha} \left( c_1(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \right) \geq 0, \]

\[ \frac{\partial}{\partial \alpha} \left( c_3(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \right) \leq 0 \]

and

\[ c_1(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \leq c_3(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \]

\[ [y(x)]^{\alpha} \] is a valid \( \alpha \) - level set.

i) If \( \left( \frac{2n-1}{2} \right) \pi x \in \left\{ \frac{2(n-1)}{2} \pi, \frac{2(n-1)+1}{2} \pi \right\}, \text{ n} = 1,3,5,..., \cos \left( \frac{2n-1}{2} \pi x \right) > 0 \). Then, for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \geq 0, \frac{\partial(c_4(\alpha))}{\partial \alpha} \leq 0 \) and \( c_2(\alpha) \leq c_4(\alpha) \), \( [y(x)]^{\alpha} \) is a valid \( \alpha \)-level set.

ii) If \( \left( \frac{2n-1}{2} \right) \pi x \in \left\{ \frac{2(n-2)+1}{2} \pi, \frac{2(n-2)+3}{2} \pi \right\}, \text{ n} = 2,4,6,..., \cos \left( \frac{2n-1}{2} \pi x \right) < 0 \). Then, for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \leq 0, \frac{\partial(c_4(\alpha))}{\partial \alpha} \geq 0 \) and \( c_2(\alpha) \geq c_4(\alpha) \), \( [y(x)]^{\alpha} \) is a valid \( \alpha \)-level set.

Consequently:

i) If \( \left( \frac{2n-1}{2} \right) \pi x \in \left\{ \frac{2(n-1)-1}{2} \pi, \frac{2(n-1)+1}{2} \pi \right\}, \text{ n} = 1,3,5,..., \) for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \geq 0, \frac{\partial(c_4(\alpha))}{\partial \alpha} \leq 0 \) and \( c_2(\alpha) \leq c_4(\alpha) \), the eigenvalues are \( \lambda_n = \frac{(2n-1)^2}{4} \pi^2 \), with associated eigenfunctions

\[ [y(x)]^{\alpha} = \left[ c_1(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \right], c_3(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \].

ii) \( \left( \frac{2n-1}{2} \right) \pi x \in \left\{ \frac{2(n-2)}{2} \pi, \frac{2(n-2)+3}{2} \pi \right\}, \text{ n} = 2,4,6,..., \) for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial(c_2(\alpha))}{\partial \alpha} \leq 0, \frac{\partial(c_4(\alpha))}{\partial \alpha} \geq 0 \) and \( c_2(\alpha) \geq c_4(\alpha) \), the eigenvalues are \( \lambda_n = \frac{(2n-1)^2}{4} \pi^2 \), with associated eigenfunctions

\[ [y(x)]^{\alpha} = \left[ c_1(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \right], c_3(\alpha) \cos \left( \frac{2n-1}{2} \pi x \right) \].

iii) If \( \alpha = 1 \), the eigenvalues are \( \lambda_n = \frac{(2n-1)^2}{4} \pi^2 \), with associated eigenfunctions

\[ \left[ y_n(x) \right]^{\alpha} = \cos \left( \frac{2n-1}{2} \pi x \right) \].

For (2,2) solution,

\[ \left[ y''_{\alpha}, \bar{y}''_{\alpha} \right] + \lambda \left[ y_{\alpha}, \bar{y}_{\alpha} \right] = 0, \]

\[ \left[ y_{\alpha}(0), y'_{\alpha}(0) \right] = 0, \left[ y_{\alpha}(1), \bar{y}_{\alpha}(1) \right] = 0 \]

is solved. Eigenvalues and eigenfunctions are same with (1,1) solution.

For (1,2) solution,

\[ \left[ y''_{\alpha}, y''_{\bar{\alpha}} \right] + \lambda \left[ y_{\alpha}, y_{\bar{\alpha}} \right] = 0 \]

\[ \left[ y'_{\alpha}(0), y'_{\bar{\alpha}}(0) \right] = 0, \left[ y_{\alpha}(1), \bar{y}_{\alpha}(1) \right] = 0 \]

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is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions \( a_4(\alpha) = a_2(\alpha) = a_3(\alpha) = 0 \), \( a_4(\alpha) \cos(k) = 0 \Rightarrow k_n = \left(\frac{2(2n-1)}{2}\right)\pi, n = 1,2,\ldots \)

The eigenvalues are \( \lambda_n = \frac{(2n-1)^2}{4}\pi^2 \), with associated eigenfunctions

\[ [y_n(x)]'' = \cos\left(\left(\frac{2n-1}{2}\right)\pi x\right).\]

For (2,1) solution,

\[ [\overline{y}_n, y_n]'' + \lambda [y_n, \overline{y}_n] = 0 \]

\( \overline{y}_n(0), y_n(0) = 0, [y_n(1), \overline{y}_n(1)] = 0. \)

Eigenvalues and eigenfunctions are same with (1,2) solution.

**Example 3.4.** If we take (3.13) as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

\[ y'(0) = 0, \quad y'(1) = 0 \]

as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

\[ [y_n'', \overline{y}_n]'' + \lambda [y_n, \overline{y}_n] = 0 \]

\( [y_n'(0), \overline{y}_n'(0)] = 0, [y_n(1), \overline{y}_n(1)] = 0. \)

is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the first boundary condition \( c_2(\alpha) = c_4(\alpha) = 0 \), from the second boundary condition,

\( y_n'(1) = -k c_4(\alpha) \sin(k) = 0, \quad \overline{y}_n'(1) = -k c_4(\alpha) \sin(k) = 0 \)

are obtained. Then, \( c_1(\alpha) \neq 0, c_3(\alpha) \neq 0, \sin(k) = 0 \Rightarrow k_n = n\pi, n = 1,2,\ldots \)

As

\( \frac{\partial (c_1(\alpha) \cos(n\pi x))}{\partial \alpha} \geq 0, \quad \frac{\partial (c_3(\alpha) \cos(n\pi x))}{\partial \alpha} \leq 0 \)

and

\( [y(\alpha)]^\alpha \) is a valid \( \alpha \)-level set.

**i) If** \( n \pi x \in \left(\frac{2(n-2)-1}{2}\pi, \frac{2(n-2)+1}{2}\pi\right) \), \( n = 2,4,6,\ldots \), \( \cos(n\pi x) > 0 \). Then, for \( c_2(\alpha), c_4(\alpha) \)

satisfying the inequality \( \frac{\partial (c_2(\alpha))}{\partial \alpha} \geq 0, \quad \frac{\partial (c_4(\alpha))}{\partial \alpha} \leq 0 \) and \( c_2(\alpha) \leq c_4(\alpha) \), \( [y(\alpha)]^\alpha \) is a valid \( \alpha \)-level set.

**ii) If** \( n \pi x \in \left(\frac{2(n-1)+1}{2}\pi, \frac{2(n-1)+3}{2}\pi\right) \), \( n = 1,3,5,\ldots \), \( \cos(n\pi x) < 0 \). Then, for \( c_2(\alpha), c_4(\alpha) \)

satisfying the inequality \( \frac{\partial (c_2(\alpha))}{\partial \alpha} \leq 0, \quad \frac{\partial (c_4(\alpha))}{\partial \alpha} \geq 0 \) and \( c_2(\alpha) \geq c_4(\alpha) \), \( [y(\alpha)]^\alpha \) is a valid \( \alpha \)-level set.

Consequently;

**i) If** \( n \pi x \in \left(\frac{2(n-2)-1}{2}\pi, \frac{2(n-2)+1}{2}\pi\right) \), \( n = 2,4,6,\ldots \), for \( c_2(\alpha), c_4(\alpha) \)

satisfying the inequality \( \frac{\partial (c_2(\alpha))}{\partial \alpha} \geq 0, \quad \frac{\partial (c_4(\alpha))}{\partial \alpha} \leq 0 \) and \( c_2(\alpha) \leq c_4(\alpha) \), the eigenvalues are \( \lambda_n = n^2\pi^2 \), with associated eigenfunctions

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ii) If \( n \pi x \in \left( \frac{2(2n - 1) + 1}{2}, \frac{2(2n - 3) + 1}{2} \right), n = 1, 3, 5, \ldots \), for \( c_2(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial (c_2(\alpha))}{\partial \alpha} \leq 0, \frac{\partial (c_4(\alpha))}{\partial \alpha} \geq 0 \) and \( c_2(\alpha) \geq c_4(\alpha) \), the eigenvalues are \( \lambda_n = n^2\pi^2 \), with associated eigenfunctions

\[ [y(x)]^n = [c_1(\alpha)\cos(n\pi x), c_2(\alpha)\cos(n\pi x)]. \]

iii) If \( \alpha = 1 \), the eigenvalues are \( \lambda_n = n^2\pi^2 \), with associated eigenfunctions

\[ [y_n(x)]^n = \cos(n \pi x). \]

For (2,2) solution,

\[ \left[ y_{a(2)}^{''}, y_{a(2)}^{'''} \right] + \lambda \left[ y_{a(2)}, y_{a(2)}^{'} \right] = 0 \]

\[ \left[ y_{a(2)}^{'}(0), y_{a(2)}^{''}(0) \right] = 0, \left[ y_{a(2)}^{''}(1), y_{a(2)}^{'}(1) \right] = 0. \]

Eigenvalues and eigenfunctions are same with (1,1) solution. For (1,2) solution,

\[ \left[ y_{a(1)}^{''}, y_{a(1)}^{'''} \right] + \lambda \left[ y_{a(1)}, y_{a(1)}^{'} \right] = 0 \]

\[ \left[ y_{a(1)}^{'}(0), y_{a(1)}^{''}(0) \right] = 0, \left[ y_{a(1)}^{''}(1), y_{a(1)}^{'}(1) \right] = 0. \]

is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions \( a_1(\alpha) = a_2(\alpha) = a_3(\alpha) = 0, a_4(\alpha)k\sin(\alpha) = 0, \)

\[ a_4(\alpha) \neq 0, \sin(\alpha) = 0 \Rightarrow k_\alpha = n\pi, n = 1, 2, \ldots \]

are obtained. The eigenvalues are \( \lambda_n = n^2\pi^2 \), with associated eigenfunctions

\[ [y_n(x)]^n = \cos(n \pi x). \]

For (2,1) solution,

\[ \left[ y_{a(2)}^{''}, y_{a(2)}^{'''} \right] + \lambda \left[ y_{a(2)}, y_{a(2)}^{'} \right] = 0 \]

\[ \left[ y_{a(2)}^{'}(0), y_{a(2)}^{''}(0) \right] = 0, \left[ y_{a(2)}^{''}(1), y_{a(2)}^{'}(1) \right] = 0. \]

Eigenvalues and eigenfunctions are same with (1,2) solution.

Example 3.5. If we take

\[ y(0) + y^{'}(0) = 0, \quad y(1) = 0 \quad (3.14) \]

as the boundary conditions of the fuzzy Sturm-Liouville problem (3.4), for (1,1) solution,

\[ \left[ y_{a(1)}^{''}, y_{a(1)}^{'''} \right] + \lambda \left[ y_{a(1)}, y_{a(1)}^{'} \right] = 0 \]

\[ \left[ y_{a(1)}^{'}(0), y_{a(1)}^{''}(0) + y_{a(1)}^{'}(0) \right] = 0, \left[ y_{a(1)}^{''}(1), y_{a(1)}^{'}(1) \right] = 0. \]

is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the boundary conditions,

\[ c_1(\alpha) + kc_2(\alpha) = 0, \quad c_3(\alpha)\cos(k) + c_5(\alpha)\sin(k) = 0 \]

\[ c_2(\alpha) + kc_4(\alpha) = 0, \quad c_3(\alpha)\cos(k) + c_4(\alpha)\sin(k) = 0 \]

are obtained. If

\[ \left| \begin{array}{cc} 1 \cos(k) & \sin(k) \end{array} \right| = 0, \]

there is the nontrivial solution of the fuzzy differential equation in (3.4) with the boundary conditions (3.14).

\[ \Rightarrow \sin(k) - k\cos(k) = 0 \Rightarrow \tan(k) = k \]

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\[ y_a(x) = c_1(\alpha)\cos(k_n x) + c_2(\alpha)\sin(k_n x), \quad \overline{y}_a(x) = c_3(\alpha)\cos(k_n x) + c_4(\alpha)\sin(k_n x) \]
\[ [y(x)]^a = \left[ y_a(x), \overline{y}_a(x) \right]. \]

For \( c_1(\alpha), c_2(\alpha), c_3(\alpha), c_4(\alpha) \) satisfying the inequality \( \frac{\partial (y_a(x))}{\partial \alpha} \geq 0, \frac{\partial (\overline{y}_a(x))}{\partial \alpha} \leq 0 \) and \( y_a(x) \leq \overline{y}_a(x) \), the eigenvalues are \( \lambda_n = k_n^2 \), with associated eigenfunctions
\[ [y(x)]^a = \left[ y_a(x), \overline{y}_a(x) \right]. \]

For (1,2) solution,
\[ \left[ y''_a, y'_a \right] + \lambda \left[ y_a, \overline{y}_a \right] = 0 \]
\[ \left[ y_a(0) + y'_a(0), \overline{y}_a(0) + \overline{y}'_a(0) \right] = 0, \left[ y_a(1), \overline{y}_a(1) \right] = 0 \]
is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions, we have
\[ a_3(\alpha) + k a_4(\alpha) = 0, \quad (1 + k) a_4(\alpha) + (1 - k) a_2(\alpha) = 0, \]
\[ a_3(\alpha)\cos(k) + a_4(\alpha)\sin(k) = 0, \quad a_1(\alpha)e^k + a_2(\alpha)e^{-k} = 0. \]

Thus, it must be \( \sin(k) - k\cos(k) = 0 \) or \( e^{2k} = \frac{1+k}{1-k} \)

If \( \sin(k) - k\cos(k) = 0 \),
\[ k_1 = 2.63008 \times 10^8, k_2 = 4.49341, k_3 = 7.72525, k_4 = 10.9041, \ldots \]
and if \( e^{2k} = \frac{1+k}{1-k} \)
\[ k_1 = 6.04989 \times 10^{-6}, k_2 = 5.05302 \times 10^{-6}, k_3 = 7.73022 \times 10^{-6}, \ldots \]
are obtained. So, if \( \sin(k) - k\cos(k) = 0 \),
\[ y_a(x) = \overline{y}_a(x) = a_3(\alpha)\sin(k_n x) + a_4(\alpha)\cos(k_n x), \]

and if \( e^{2k} = \frac{1+k}{1-k} \)
\[ y_a(x) = -a_1(\alpha)e^{k_n x} - a_2(\alpha)e^{-k_n x}, \overline{y}_a(x) = a_3(\alpha)e^{k_n x} + a_2(\alpha)e^{-k_n x}. \]

Consequently,
\[ i) \text{ if } \sin(k) - k\cos(k) = 0, \text{ the eigenvalues are } \lambda_n = k_n^2, \text{ with associated eigenfunctions} \]
\[ y_n(x) = a_3(\alpha)\sin(k_n x) + a_4(\alpha)\cos(k_n x). \]
\[ \frac{\partial (y_a(x))}{\partial \alpha} \leq 0 \text{ and } y_a(x) \leq \overline{y}_a(x), \text{ the eigenvalues are } \lambda_n = k_n^2, \text{ with associated eigenfunctions} \]
\[ y_a(x) = -a_1(\alpha)e^{k_n x} - a_2(\alpha)e^{-k_n x}, \overline{y}_a(x) = a_3(\alpha)e^{k_n x} + a_2(\alpha)e^{-k_n x}. \]

For (2,2) solution,
\[ \left[ y''_a, y'_a \right] + \lambda \left[ y_a, \overline{y}_a \right] = 0 \]
\[ \left[ y_a(0) + y'_a(0), \overline{y}_a(0) + \overline{y}'_a(0) \right] = 0, \left[ y_a(1), \overline{y}_a(1) \right] = 0 \]
is solved. The solution of the fuzzy differential equation is (3.5)-(3.7). From the boundary conditions, we have
\[ \frac{\sin(k)}{k} c_1(\alpha) + k\cos(k)c_2(\alpha) = 0, \quad \frac{\sin(k)}{k} c_3(\alpha) + k\cos(k)c_4(\alpha) = 0, \]
\[ \cos(k)c_1(\alpha) + \sin(k)c_2(\alpha) = 0, \quad \cos(k)c_3(\alpha) + \sin(k)c_4(\alpha) = 0. \]
So, from these equation systems, it must be
\[
\frac{\sin^2(k)}{k} - k \cos^2(k) = 0 \Rightarrow \tan^2(k) = k^2.
\]

\[k_1 = 3.51833 \times 10^{-8}, \quad k_2 = 2.02876, \quad k_3 = 4.91318, \ldots\]

Thus, for \(c_1(a), c_2(a), c_3(a), c_4(a)\) satisfying the inequality \(\frac{d(\gamma_n(x))}{da} \geq 0\), \(\frac{d(\overline{\gamma}_n(x))}{da} \leq 0\) and \(\gamma_n(x) \leq \overline{\gamma}_n(x)\), the eigenvalues are \(\lambda_n = k_n^2\), with associated eigenfunctions
\[
\gamma_n(x) = c_1(a) \cos(k_n x) + c_2(a) \sin(k_n x), \quad \overline{\gamma}_n(x) = c_3(a) \cos(k_n x) + c_4(a) \sin(k_n x)
\]
\[
[y_n(x)]^a = \left[\gamma_n(x), \overline{\gamma}_n(x)\right].
\]

For (2.1) solution,
\[
\begin{bmatrix}
\gamma'_a, \gamma''_a \\
\gamma'_a(0) + \gamma''_a(0), \gamma'_a(0) + \gamma''_a(0)
\end{bmatrix} = 0
\]
\[
\left[\gamma'_a(1), \gamma''_a(1) \right] = 0
\]
is solved. The solution of the fuzzy differential equation is (3.8)-(3.10). From the boundary conditions, we have
\[
a_3(a) + ka_1(a) = 0, \quad (1 - k)a_4(a) + (1 + k)a_2(a) = 0,
\]
\[
a_3(a) \cos(k) + a_4(a) \sin(k) = 0, \quad a_1(a) e^k + a_2(a) e^{-k} = 0.
\]

Thus, it must be
\[
\sin(k) - k \cos(k) = 0 \text{ or } e^{2k} = \frac{1-k}{1+k}\]

If \(\sin(k) - k \cos(k) = 0\),
\[
k_1 = 2.63008 \times 10^8, \quad k_2 = 4.49341, \quad k_3 = 7.72525, \quad k_4 = 10.9041, \ldots
\]

and if \(e^{2k} = \frac{1-k}{1+k}\)
\[
k_1 = 3.98993 \times 10^{-17}, \quad k_2 = 2.5154 \times 10^{-17}, \quad k_3 = 9.54151 \times 10^{-18}, \ldots
\]

are obtained. So, if \(\sin(k) - k \cos(k) = 0\),
\[
\gamma_n(x) = \overline{\gamma}_n(x) = a_3(a) \sin(k_n x) + a_4(a) \cos(k_n x),
\]

and if \(e^{2k} = \frac{1+k}{1-k}\)
\[
y_n(x) = a_1(a) e^{k_n x} - a_2(a) e^{-k_n x}, \quad \overline{\gamma}_n(x) = a_1(a) e^{k_n x} + a_2(a) e^{-k_n x},
\]
\[
[y_n(x)]^a = \left[\gamma_n(x), \overline{\gamma}_n(x)\right].
\]

Consequently,
\[\text{i) if } \sin(k) - k \cos(k) = 0, \text{ the eigenvalues are } \lambda_n = k_n^2, \text{ with associated eigenfunctions}
\]
\[y_n(x) = a_3(a) \sin(k_n x) + a_4(a) \cos(k_n x),
\]
\[\overline{\gamma}_n(x) = a_1(a) e^{k_n x} - a_2(a) e^{-k_n x}, \quad \gamma_n(x) = a_1(a) e^{k_n x} + a_2(a) e^{-k_n x},\]
\[
[y_n(x)]^a = \left[\gamma_n(x), \overline{\gamma}_n(x)\right].
\]

\[\text{References}
\]


