**Arens Regularity of Bilinear Mapping and Reflexivity**

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**Abstract:** Let $X$, $Y$ and $Z$ be normed spaces. In this article we give a tool to investigate Arens regularity of a bounded bilinear map $f : X \times Y \to Z$. Also, under some assumptions on $f^{***}$ and $f^{****}$, we give some new results to determine reflexivity of the spaces.

**Keywords:** Arens regular, bilinear map, topological center, factor, second dual.

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**INTRODUCTION AND PRELIMINARIES**

Arens showed in [1] that a bounded bilinear map $f : X \times Y \to Z$ on normed spaces, has two natural different extensions $f^{***}$, $f^{****}$ from $X^{**} \times Y^{**}$ into $Z^{**}$. When these extensions are equal, $f$ is said to be Arens regular. Throughout the article, we identify a normed space with its canonical image in the second dual.

We denote by $X^{*}$ the topological dual of a normed space $X$. We write $X^{**}$ and so on. Let $X$, $Y$ and $Z$ be normed spaces and $f : X \times Y \to Z$ be a bounded bilinear mapping. The natural extensions of $f$ are as following:

(i) $f^{*} : X^{*} \times Y \to Y^{*}$, give by $<f^{*}(x^{*}), y> = f(x, y)$ where $x \in X$, $y \in Y$, $x^{*} \in X^{*}$ ($f^{*}$ is said the adjoint of $f$).

(ii) $f^{**} = (f^{*})^{*} : Y^{**} \times Z \to X^{*}$, give by $<f^{**}(y^{**}, z^{*}), x^{*}> = <y^{**}, f^{*}(z^{*}, x^{*})>$. 

(iii) $f^{***} = (f^{**})^{*} : X^{**} \times Y^{**} \to Z^{*}$, give by $<f^{***}(x^{**}, y^{**}), z^{*}> = <x^{**}, f^{**}(y^{**}, z^{*})>$.  

Let now $f^{+} : Y \times X \to Z$ be the flip of $f$ defined by $f^{+}(y, x) = f(x, y)$, for every $x \in X$ and $y \in Y$. Then $f^{+}$ is a bounded bilinear map and it may extends as above to $f^{++} : Y^{**} \times X^{**} \to Z^{**}$. In general, the mapping $f^{+++} : X^{**} \times Y^{**} \to Z^{**}$ is not equal to $f^{***}$. When these extensions are equal, then $f$ is Arens regular.

One may define similarly the mappings $f^{****} : Z^{**} \times X^{**} \to Y^{***}$ and $f^{*****} : Y^{***} \times Z^{**} \to X^{***}$ and the higher rank adjoints. Consider the nets $(x_{a}) \subseteq X$ and $(y_{\beta}) \subseteq Y$ converge to $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$ in the weak* –topologies, respectively, then

$f^{***}(x^{**}, y^{**}) = w^{*} - \lim_{a} w^{*} - \lim_{\beta} f(x_{a}, y_{\beta})$  
and  
$f^{****}(x^{**}, y^{**}) = w^{*} - \lim_{\beta} w^{*} - \lim_{a} f(x_{a}, y_{\beta})$

So Arens regularity of $f$ is equivalent to the following

$\lim_{a} \lim_{\beta} <z^{*}, f(x_{a}, y_{\beta})> = \lim_{\beta} \lim_{a} <z^{*}, f(x_{a}, y_{\beta})>$

If the limits exit for each $z^{*} \in Z^{*}$. The map $f^{***}$ is the unique extension of $f$ such that $x^{**} \to f^{***}(x^{**}, y^{**}) : X^{**} \to Z^{**}$ is weak* – weak* continuous for each $y^{**} \in Y^{**}$ and $y^{**} \to f^{***}(x^{**}, y^{**}) : Y^{**} \to Z^{**}$ is weak* – weak* continuous for each $x^{**} \in X^{**}$. The left topological center of $f$ is defined by

$Z_{L}(f) = \{x^{**} \in X^{**} : y^{**} \to f^{***}(x^{**}, y^{**}) : Y^{**} \to Z^{**} \text{ is weak* – weak* continuous}\}.$

Since $f^{++} : X^{**} \times Y^{**} \to Z^{**}$ is the unique extension of $f$ such that the map $y^{**} \to f^{++}(x^{**}, y^{**}) : Y^{**} \to Z^{**}$ is weak* – weak* continuous for each $x^{**} \in X^{**}$, we can set

$Z_{r}(f) = \{y^{**} \in Y^{**} : x^{**} \to f^{+++}(x^{**}, y^{**}) : X^{**} \to Z^{**} \text{ is weak* – weak* continuous}\}.$

Again since the map $x^{**} \to f^{+++}(x^{**}, y^{**}) : X^{**} \to Z^{**}$ is weak* – weak* continuous for each $y^{**} \in Y^{**}$, we have

$Z_{r}(f) = \{y^{**} \in Y^{**} : x^{**} \to f^{+++}(x^{**}, y^{**}) : (x^{**} \in X^{**})\}.$
A bounded bilinear mapping $f$ is Arens regular if and only if $Z(f) = X^\ast\ast$ or equivalently $Z_r(f) = Y^\ast$. It is clear that $X \subseteq Z(f)$. If $X = Z(f)$ then the map $f$ is said to be left strongly irregular. Also $Y \subseteq Z_r(f)$ and if $Y = Z_r(f)$ then the map $f$ is said to be right strongly irregular. A bounded bilinear mapping $f: X \times Y \to Z$ is said to factor if it is onto.

**Investigate Arens regularity of bounded bilinear maps**

S. Mohammadzadeh and Vishki H.R proved in [6] acriterion concerning to the regularity of a bounded bilinear map. They showed that $f$ is Arens regular if and only if $f^{++++}(Z^\ast, X^\ast) \subseteq Y^\ast$. In the section we provide the same conditions of Arens regularity. First, we give a similar lemma to the [6, Theorem 2.1].

**Lemma 2.1.** For a bounded bilinear map $f$ from $X \times Y$ into $Z$ the following statements are equivalent:

(i) $f$ is Arens regular;
(ii) $f^{++++} = f^{++++}$;
(iii) $f^{++++} = f^{++++}$.

**Proof.** If (i) hold then $f^T$ is Arens regular. Therefor $f^{++++} = f^{++++}$. For every $x^\ast \in X^\ast$, $y^\ast \in Y^\ast$ and $z^\ast \in Z^\ast$ we have

$$\langle f^{++++}(y^\ast, z^\ast), x^\ast \rangle = \langle z^\ast, f^{++++}(y^\ast, x^\ast) \rangle = \langle f^{++++}(z^\ast, x^\ast), y^\ast \rangle = \langle f^{++++}(z^\ast, y^\ast), x^\ast \rangle.$$

Therefore $f^{++++} = f^{++++}$.

(ii) $\Rightarrow$ (iii) Let $x^\ast \in X^\ast$, $y^\ast \in Y^\ast$ and $z^\ast \in Z^\ast$ we have

$$\langle f^{++++}(z^\ast, x^\ast), y^\ast \rangle = \langle z^\ast, f^{++++}(x^\ast, y^\ast) \rangle = \langle x^\ast, f^{++++}(z^\ast, y^\ast) \rangle = \langle x^\ast, f^{++++}(z^\ast, x^\ast) \rangle.$$

(iii) $\Rightarrow$ (i) Let $x^\ast \in X^\ast$, $y^\ast \in Y^\ast$ and $z^\ast \in Z^\ast$ we have

$$\langle f^{++++}(x^\ast, y^\ast), z^\ast \rangle = \langle f^{++++}(y^\ast, x^\ast), z^\ast \rangle = \langle f^{++++}(z^\ast, x^\ast), y^\ast \rangle = \langle f^{++++}(z^\ast, y^\ast), x^\ast \rangle = \langle f^{++++}(x^\ast, y^\ast), z^\ast \rangle = \langle f^{++++}(x^\ast, y^\ast), z^\ast \rangle.$$

It follows that $f$ is Arens regular and this completes the proof.

**Theorem 2.2.** Bounded bilinear map $f$ from $X \times Y$ into $Z$ is Arens regular if and only if $f^{++++}(Y^\ast, Z^\ast) \subseteq X^\ast$.

**Proof.** Let $y^\ast \in Y^\ast$ and $z^\ast \in Z^\ast$ be arbitrary. If $f$ is Arens regular Then $f^{++++} = f^{++++}$. Therefore

$$f^{++++}(y^\ast, z^\ast) = f^{++++}(y^\ast, z^\ast) = f^{++++}(y^\ast, z^\ast) = f^{++++}(y^\ast, z^\ast) \subseteq X^\ast.$$

Conversely, suppose $f^{++++}(Y^\ast, Z^\ast) \subseteq X^\ast$. Let $(x_\alpha, y_\beta) \subseteq X$ and $(y_\beta, x_\alpha) \subseteq Y$ be two nets that are converge to $x^\ast$ and $y^\ast$ in the weak-topologies, respectivety. Then

$$\langle f^{++++}(x_\alpha, y_\beta), z^\ast \rangle = \lim_{\alpha, \beta} \langle f^{++++}(x_\alpha, y_\beta), z^\ast \rangle = \lim_{\alpha, \beta} \langle f^{++++}(x_\alpha, y_\beta), z^\ast \rangle = \lim_{\alpha, \beta} \langle f^{++++}(x_\alpha, z^\ast), y_\beta \rangle = \lim_{\alpha, \beta} \langle x_\alpha, f^{+++}(z^\ast, y_\beta) \rangle = \lim_{\alpha, \beta} \langle x_\alpha, f^{+++}(z^\ast, y_\beta) \rangle = \lim_{\alpha, \beta} \langle f^{+++}(x_\alpha, y_\beta), z^\ast \rangle.$$

Therefore $f$ is Arens regular and this completes the proof.

**Corollary 2.3.** For a bounded bilinear map $f: X \times Y \to Z$, the following statements are equivalent:

(i) $f^{++++}(Y^\ast, Z^\ast) \subseteq X^\ast$;
(ii) $f$ and $f^*$ are Arens regular;
(iii) $f^{++++} = f^{++++}$.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows from the fact that $f^{++++}(Y^\ast, Z^\ast) \subseteq f^{++++}(Y^\ast, Z^\ast) \subseteq X^\ast$. Now Theorem 2.2 implies the Arens regularity of $f$, or equivalently $f^{++++} = f^{++++}$. From which $f^{++++}(Z^\ast, Y^\ast) = f^{++++}(Z^\ast, Y^\ast) = f^{++++}(Z^\ast, Y^\ast) = f^{++++}(Y^\ast, Z^\ast) \subseteq X^\ast$. 

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Therefore the Arens regularity of $f^{**}$ follows again by Theorem 2.2. Thus $f^{**}$ is Arens regular.

(ii) $\Rightarrow$ (iii) If $f$ is Arens regular. Then

$$f^{****} = f^{*****} \Rightarrow f^{****} = f^{*******}$$

(2-1)

Now if $f$ is Arens regular. Then we have

$$f^{****} = f^{*******}$$

(2-2)

The equalities (2-1) and (2-2) together establish the assertion.

(iii) $\Rightarrow$ (i) First we show that $f^{*******} = f^{*****}$. For every $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{***} \in Z^{***}$

$$< f^{*****}(y^{**}, z^{***}), x^{**} >= < f^{******}(z^{***}, y^{**}), x^{**} >= < f^{******}(y^{**}, x^{**}) >$$

$$= < z^{***}, f^{*****}(x^{**}, y^{**}) >= < f^{******}(z^{***}, x^{**}), y^{**} >=$$

$$= < f^{******}(z^{***}, x^{**}), y^{**} >= < f^{******}(y^{**}, z^{***}), y^{**} >=$$

$$= < x^{**}, f^{*****}(z^{***}, y^{**}) >= < f^{******}(y^{**}, z^{***}), x^{**} >$$

As $f^{******}(Y^{**}, Z^{***})$ lies in $X^{*}$ thus $f^{*******}(Y^{**}, Z^{***}) \subseteq X^{*}$ and the proof.

**Theorem 2.4.** Let $X$ and $A$ be normed spaces and $g : X \times A \rightarrow X$ is a bounded bilinear map. If $g^{***} : X^{**} \times A^{**} \rightarrow X^{*}$ factor and $g$ is Arens regular. Then $g$ is Arens regular.

**Proof.** Let $g^{***}$ factor. Thus for every $x^{**} \in X^{**}$ there exists $y^{**} \in X^{**}$ and $b^{**} \in A^{**}$ such that $x^{**} = g^{***}(y^{**}, b^{**})$. Suppose that $a^{**} \in A^{**}$ and $(a_{\beta}, y_{\beta}) \subseteq A^{**} \subseteq Y^{**}$ be bounded nets weak*–converging to $a^{**}, b^{**}$ and $y^{**}$ respectively. For every $x^{*} \in X^{*}$ we have

$$< g^{********}(x^{*}, a^{**}), x^{*} >= < g^{********}(x^{*}, a^{**}), x^{*} >$$

$$= < g^{********}(x^{*}, a^{**}), g^{**}(y^{**}, b^{**}) >= < g^{**}(g^{********}(x^{*}, a^{**}), y^{**}), b^{**} >$$

$$= < g^{********}(x^{*}, a^{**}), y^{**} > = \lim_{\beta} < g^{********}(x^{*}, a^{**}), y^{**} >$$

$$= \lim_{\beta} < g^{***}(x^{*}, a^{**}), y_{\beta} > = \lim_{\beta} < g^{**}(a_{\beta}, y_{\beta}), x^{*} >$$

$$= \lim_{\beta} < g^{***}(a_{\beta}, y_{\beta}), x^{*} > = \lim_{\beta} < a^{***}, g^{********}(b_{\beta}, y_{\beta}), x^{*} >$$

$$= \lim_{\beta} \lim_{\gamma} < g^{********}(b_{\beta}, y_{\beta}), x^{*}, a_{\alpha} > = \lim_{\beta} \lim_{\gamma} < g^{********}(b_{\beta}, y_{\beta}), g^{********}(x^{*}, a_{\alpha}) >$$

$$= \lim_{\beta} \lim_{\gamma} < g^{********}(b_{\beta}, y_{\beta}), g^{********}(x^{*}, a_{\alpha}) > = \lim_{\beta} \lim_{\gamma} < g^{********}(b_{\beta}, y_{\beta}), g^{********}(x^{*}, a_{\alpha}) >$$

$$= \lim_{\beta} \lim_{\gamma} < g^{********}(b_{\beta}, y_{\beta}), g^{********}(x^{*}, a_{\alpha}) > = \lim_{\beta} \lim_{\gamma} < g^{********}(b_{\beta}, y_{\beta}), g^{********}(x^{*}, a_{\alpha}) >$$

As $g^{********}(Y^{**}, Z^{***})$, lies in $X^{*}$ thus $g^{***********}(Y^{**}, Z^{***}) \subseteq X^{*}$ and the proof.

**Theorem 3.1.** For a bounded bilinear map $f : X \times Y \rightarrow Z$.
(i) If \( f^{****} \) factor then both \( f \) and \( f^* \) are Arens regular if and only if \( Y \) is reflexive.
(ii) If \( f^{******} \) factor then both \( f \) and \( f^* \) are Arens regular if and only if \( X \) is reflexive.

**Proof.** We only give a proof for (ii). A similar proof applies for (i). Let \( f \) and \( f^* \) are Arens regular by Corollary 2.3 \( f^{******}(Y^{***}, Z^{****}) \subseteq X^* \). On the other hand \( f^{******} \) factors, So \( f^{******}(Y^{**} \times Z^{***}) = X^{***} \). Therefore \( X^{***} \subseteq X^* \). Conversely, using [8,2,3] is obvious.

As an immediate consequence of Theorem 3.1 and [8,2,4], we have the next Corollary.

**Corollary 3.2.** If one of the two following statement is assumed:

(i) \( f \) and \( f^* \) are Arens regular and \( f^{******} \) factor;
(ii) \( f \) and \( f^{**} \) are Arens regular and \( f^{****} \) factor;
Then every adjoint map and every flip map of \( f \) is Arens regular.

**Corollary 3.3.** Let \( f \) and \( f^* \) are Arens regular and \( f^{******} \) factor. Then \( f \) is left strongly irregular if and only it is right strongly irregular.

**Proof.** The follows by applying Theorem 3.1 and [8,Theorem 2.5].

If \( X \) is reflexive. Then obviously bounded bilinear map \( f \) from \( X \times Y \) into \( Z \) is Arens regular. But from Arens regularity \( f \) does not imply the reflexivity of \( X \). The next Theorem, we use the Theorem 2.2 and show that if \( f^{**}(z^*Y) = X^* \). Then \( X \) is reflexive.

**Theorem 3.4.** Let bounded bilinear map \( f \) from \( X \times Y \) into \( Z \) is Arens regular and let \( Y \) is a Banach space. If \( f^{**}(z^*Y) = X^* \) for some \( z^* \in Z^* \); Then \( X \) is reflexive.

**Proof.** Let \( h : Y \rightarrow X^* \) define by \( h(y) = f^{**}(z^*, y) \) for every \( y \in Y \). Obviously \( h^*(x^*) = f^{**}(x^*, z^*) \) for every \( x^* \in X^{**} \). We have

\[
< h^*(y^*), x^* > = < y^*, h^*(x^*) > = < y^*, f^{**}(x^*, z^*) > =\]

\[
= < f^{******}(y^*, x^*), z^* > = < f^{******}(z^*, y^*), x^* > = < f^{******}(y^*, z^*), x^* >.
\]

Therefore \( h^*(y^*) = f^{******}(y^*, z^*) \) for every \( y^* \in Y^{**} \). Now Theorem 2.2 implies that \( f^{******}(Y^{**}, Z^*) \subseteq X^* \). Since \( f^{**}(z^*, Y) = X^* \) thus \( h \) is onto. Therefore \( h^* \) from \( Y^{**} \) into \( X^{***} \) is onto. Let \( x^{***} \in X^{***} \) so there exists \( y^{**} \in Y^{**} \) such that \( x^{***} = h^*(y^{**}) = f^{******}(y^*, z^*) \in X^* \). Thus \( X \) is reflexive.

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