Boundedness of Fractional Integral Operators with Variable Kernels Associate to Variable Exponents

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Abstract: Let $\alpha(\cdot)$ satisfy the log-Hölder continuity condition and $1 < \alpha(\cdot) < n$. Suppose $T_{\Omega,\alpha(\cdot)}$ is the fractional integral operator with variable kernel associate to variable exponent. In this paper, using the properties of weighted Morrey spaces, we prove that $T_{\Omega,\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\omega^{p(\cdot)};\omega^{q(\cdot)})$ to $L^{p(\cdot)}(\omega^{q(\cdot)})$.

Keywords: fractional integral operator; variable exponent; variable kernel; weighted Morrey space.

INTRODUCTION

Suppose that $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $n > 2$ and $d\sigma$ is the normalized Lebesgue measure on $S^{n-1}$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times S^{n-1}$ is said to belong to $L^r(\mathbb{R}^n) \times L^s(S^{n-1}), r \geq 1$, if it satisfies the following conditions.

(i) $\Omega(x,z) = \Omega(x,z)$, for any $\lambda > 0$ and $x,z \in \mathbb{R}^n$;

(ii) $\int_{\mathbb{R}^n} \int_{S^{n-1}} \Omega(x,z)' |d\sigma(z')|^\nu < \infty$;

(iii) for any $x \in \mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x,z)' |d\sigma(z')| = 0$, where $z' = \frac{z}{|z|}$ and $z \neq 0$.

Assume $\Omega \in L^r(\mathbb{R}^n) \times L^s(S^{n-1}), r \geq 1$. We say that $\Omega$ satisfies the $L^1$-Dini condition if the conditions (i),(ii),(iii) above hold and $\int_0^1 \omega(t) dt < \infty$, where $\omega(t)$ is defined by

$$\omega(t) = \sup_{x \in \mathbb{R}^n} \left( \int_{S^{n-1}} |\Omega(x,\rho x') - \Omega(x,x')|^\nu |d\sigma(x')| \right)^{\frac{1}{\nu}},$$

which $\rho$ is a rotation in $\mathbb{R}^n$ and $\|\rho\| = \sup_{x \in \mathbb{R}^n} |\rho x - x|$.

A measurable function $\alpha(\cdot)$ is called a variable exponent if $\alpha(\cdot) : \mathbb{R}^n \rightarrow (0,\infty)$. For a measurable subset $G \subset \mathbb{R}^n$, we write $\alpha_\infty = \inf_{x \in G} \alpha(x)$, $\alpha_\infty = \sup_{x \in G} \alpha(x)$. Let $P(\mathbb{R}^n)$ denote the set of functions $\alpha(\cdot)$ satisfying $1 < \alpha_\infty = \alpha_\infty < n$.

We say that $\alpha(\cdot)$ satisfies the log-Hölder continuity condition, if

$$|\alpha(x) - \alpha(y)| \leq \frac{C}{\log(1/|x - y|)}; \quad |x - y| \leq \frac{1}{2};$$

$$|\alpha(x) - \alpha(y)| \leq \frac{C}{\log(e^{+}|x|)}; \quad |y| \geq |x|.$$

Set $\Omega(x,z) \in L^r(\mathbb{R}^n) \times L^s(S^{n-1}), r \geq 1$, satisfying the $L^1$-Dini condition. The fractional integral operator with variable kernels associate to variable exponents is defined by

$$T_{\Omega,\alpha(\cdot)} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n-\alpha(\cdot)}} f(y) dy.$$
where \( \alpha(\cdot) \) is a variable exponent, satisfying the log-Hölder continuity condition and \( \alpha(\cdot) \in \mathcal{P}(\mathbb{R}^+) \).

Obviously, if \( \alpha(\cdot) = \alpha \) being a constant, and \( \Omega = 1 \), the fractional integral operator with variable kernels associate to variable exponents is become to the classical fractional integral \( T_{\alpha,\omega} \). In 1974, Muckenhoupt and Wheeden[1] studied the boundedness of \( T_{\alpha,\omega} \) on Lebesgue spaces for \( 0 < \alpha < n \). In 2009, Komori and Shirai[2] defined the weighted Morrey space, which is a generalized weighted Lebesgue space. Then, the boundedness of fractional integral operators and fractional maximal operators, and their commutators on weighted Morrey spaces were discussed by Wang[3].

In this paper, we discussed the fractional integral operators with variable kernels associate to variable exponents on weighted Morrey spaces.

**PRELIMINARIES**

We recall several useful lemmas and definitions.

**Lemma 2.1** (see [4] and [5]). Let \( \alpha(\cdot) \) satisfy log-Hölder condition and \( \alpha(\cdot) \in \mathcal{P} \). Suppose that 
\[ \Omega \in L^r(\mathbb{R}^s) \times L^r(S^{n-1}), \ r \geq \frac{n}{n-\alpha(\cdot)}, \ \text{satisfying the} \ \text{L}^r-\text{Dini condition.} \]
Set \( 1/q = 1/p - \alpha(\cdot)/n \). Then there exists a positive constant \( C \), such that for all \( f \in L_p(\Omega) \),
\[ \left\| T_{\alpha,\omega} f \right\|_{L^p} \leq C \left\| f \right\|_{L^p}. \]

**Lemma 2.2** (see [6]). Let \( \omega \in A_p \). Then for any ball \( B \), there exists a constant \( C > 0 \), such that \( \omega(2B) \leq C \omega(B) \). In fact, \( \omega(\lambda B) \leq C \lambda^{n/p} \omega(B) \) for \( \lambda > 1 \), where the constant \( C \) is independent of \( B \) and \( \lambda \).

Let \( 1 < p < q < \infty \). We say that a weighted function \( \omega \) is belong to weighted set \( A(p,q) \), if for any ball \( B \subset \mathbb{R}^n \), there exists a constant \( C > 0 \) independent of \( B \), such that
\[ \left( \frac{1}{|B|} \int_B \omega(x)^q \, dx \right)^{1/q} \left( \frac{1}{|B|} \int_B \omega(x)^p \, dx \right)^{1/p} \leq C. \]

We say a weighted function belong to the inverse Hölder inequality \( RH_{\omega} \), if there exist constants \( s > 1 \) and \( C > 0 \), such that
\[ \left( \frac{1}{|B|} \int_B \omega(x)^s \, dx \right)^{1/s} \leq C \left( \frac{1}{|B|} \int_B \omega(x) \, dx \right). \]

As we all know that if \( \omega \in A_p \), then for all \( s > p \), \( \omega \in A_s \). If \( \omega \in A_p \), then there exists \( s > 1 \), such that \( \omega \in RH_{\omega} \).

**Lemma 2.3** (see [7]). Let \( \omega \in RH_{\omega} \), \( s > 1 \). Then there exists a positive constant \( C \), such that for any measurable subset \( E \subset B \),
\[ \frac{\omega(E)}{\omega(B)} \leq C \left( \frac{|E|}{|B|} \right)^{(s-1)/s}. \]

**Definition 2.4** (see [2]). Let \( 1 \leq p < \infty \), \( 0 < \kappa < 1 \) and \( \omega \) be a weighted function. Then a weighted Morrey space \( L^{p,\kappa}(\omega) \) is defined by
\[ L^{p,\kappa}(\omega) := \left\{ f \in L^{p,\kappa}_w(\omega) : \left\| f \right\|_{L^{p,\kappa}(\omega)} < \infty \right\}, \]
where \( \left\| \cdot \right\|_{L^{p,\kappa}(\omega)} := \sup_{B} \left( \frac{1}{|B|^\kappa} \int_B f(x)^p \omega(x) \, dx \right)^{1/p}. \)

**Definition 2.5** (see [2]). Let \( 1 \leq p < \infty \), \( 0 < \kappa < 1 \), \( u \) and \( v \) be weighted functions. Then a weighted Morrey space \( L^{p,\kappa}(u,v) \) is defined by
\[ L^{p,\kappa}(u,v) := \left\{ f \in L^{p,\kappa}_w(u,v) : \left\| f \right\|_{L^{p,\kappa}(u,v)} < \infty \right\}, \]
where \( \left\| \cdot \right\|_{L^{p,\kappa}(u,v)} := \sup_{B} \left( \frac{1}{|B|^\kappa} \int_B f(x)^p u(x) \, dx \right)^{1/p}. \)
Lemma 2.6 (see [8] and [9]). Let \( \alpha() \in \mathcal{P} \). If \( \alpha() \) is log-Hölder continuous at origin, then
\[
C^{-1} |x|^{\mu(0)} \leq |x|^{\mu(0)} \leq C |x|^{\mu(0)}^\ast, |x| < 1.\]
If \( \alpha() \) is log-Hölder continuous at the infinity, then
\[
C^{-1} |x|^{\mu(0)} \leq |x|^{\mu(0)} \leq C |x|^{\mu(0)}^\ast, |x| \geq 1, \text{ where } \alpha(\infty) = \lim_{x \to \infty} \alpha(x).\]

**THE MAIN RESULT**

The main result of this paper is as follows.

**Theorem 3.1** Set \( \Omega \in L^r(\mathbb{R}^r) \times L^r(\mathbb{R}^r \times \cdots \times \mathbb{R}^r), r \in (1, \infty) \). Let \( \alpha() \) be a variable exponent satisfying the log-Hölder continuity condition. Suppose \( r' < p < n/\alpha() \), \( 1/q = 1/p - \alpha()/\alpha() \), \( 0 < \kappa < p/q \) and \( \omega' \in A(p/r', q/r') \). Then
\[
\left\| f_{\Omega, \omega()} \right\|_{L^\omega(\omega, \omega')} \leq C \left\| f \right\|_{L^\omega(\omega, \omega')}.
\]

**Proof:** Fix a ball \( B = B(x_0, r_B) \subset \mathbb{R}^n \). Let \( f = f_1 + f_2 \), where \( f_1 = f_{2 \theta}, \chi_{2B} \) denoting the characteristic function of \( 2B \).

Since \( T_{\Omega, \omega()}(f) \) is a linear operator, we write
\[
\frac{1}{\omega^\mu(B)} \left( \int_{2\theta} |T_{\Omega, \omega()}(f)(x)|^\mu \omega(x) dx \right)^{\frac{1}{\mu}}
\]
\[
\leq C \left\| f \right\|_{L^\omega(\omega', \omega)} \left( \int_{2\theta} \omega(x) dx \right)^{\frac{1}{\mu}}
\]
\[
= I_1 + I_2.
\]

Set \( p_1 = p/r', q_1 = q/r' \) and \( \nu = \omega' \). Since \( \nu \in A(p_1, q_1) \), we can get (see [1])
\[
\nu^\mu = \omega' \in A(p_1/r', p_1).
\]

By Lemma 2.1 and Lemma 2.2, we have
\[
I_1 \leq C \frac{1}{\omega^\mu(B)} \left( \int_{2\theta} |f(x)|^\mu \omega(x) dx \right)^{\frac{1}{\mu}}
\]
\[
\leq C \left\| f \right\|_{L^\omega(\omega', \omega)} \left( \int_{2\theta} \omega(x) dx \right)^{\frac{1}{\mu}}
\]
\[
\leq C \left\| f \right\|_{L^\omega(\omega', \omega)}.
\]

To estimate \( I_2 \), using the Hölder inequality, we obtain
\[
\left| T_{\Omega, \omega()}(f)(x) \right| \leq \int_{2\theta} \Omega(x, x-y) |f(y)| dy
\]
\[
\leq \sum_{i=1}^n \left( \int_{2\theta} \Omega(x, x-y) dy \right)^{\frac{1}{p'}} \left( \int_{2\theta} |f(y)|^{p'\omega(x-y)} dy \right)^{\frac{1}{p'}}.
\]

Since \( x \in B, y \in 2^{k+1}B \setminus 2^k B \), we see that \( |x-y| \sim |x_0-y| \sim 2^{k+1}r_B \), where \( x_0 \) is the center of \( B \).

Hence,
\[
\left( \int_{2\theta} \Omega(x, x-y) dy \right)^{\frac{1}{p'}} \leq C \left\| \Omega \right\|_{L(\mathbb{R}^n)} B^\nu.
\]

By Lemma 2.6, if \( 2^{k+1}r_B \leq 1 \), then
\[
\left( \int_{2\theta} |f(y)|^{p'\omega(x-y)} dy \right)^{\frac{1}{p'}} \leq C \left\| f \right\|_{L^\omega(\omega', \omega)} B^\nu.
\]

If \( 2^{k+1}r_B > 1 \), then
\[
\left( \int_{2\theta} |f(y)|^{p'\omega(x-y)} dy \right)^{\frac{1}{p'}} \leq C \left\| f \right\|_{L^\omega(\omega', \omega)} B^\nu.
\]

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Here and below we only prove the case that \(2^{k+1}r_k \leq 1\). The other one is similar and simple. (2), (3) and (4) tell us that
\[
|I_{\Omega_1,\omega_1}(f)(x)| \leq C \|f\|_{L_\nu^{\alpha}(\nu(x))} \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \left( \int_{2^{k+1}B} |f(y)|^{\nu} \, dy \right)^{1/\nu}.
\]
By the Hölder inequality and the definition of \(\nu \in A(p_1,q_1)\), we have
\[
\left( \int_{2^{k+1}B} |f(y)|^{\nu} \, dy \right)^{1/\nu} \leq \left( \int_{2^{k+1}B} |f(y)|^{\nu} \nu(y)^{\nu} \, dy \right)^{1/(p_1\nu)} \left( \int_{2^{k+1}B} \nu(y)^{\nu} \, dy \right)^{(1/\nu)}
\]
\[
\leq C \left( \int_{2^{k+1}B} |f(y)|^{\nu} \, dy \right)^{1/p} \left( \frac{2^{k+1}B}{\nu(y)(2^{k+1}B)^{\nu(y)}} \right)^{1/\nu}
\]
\[
\leq C \left| \int f \right|_{L_\nu^{\alpha}(\nu(x),\nu')} \omega^\alpha \left( 2^{k+1}B \right)^{\nu/\nu'} \omega^\alpha \left( 2^{k+1}B \right)^{\nu/\nu'}
\]
\[
\leq C \left| \int f \right|_{L_\nu^{\alpha}(\nu(x),\nu')} \omega^\alpha \left( 2^{k+1}B \right)^{\nu/\nu'} \omega^\alpha \left( 2^{k+1}B \right)^{\nu/\nu'}
\]
Thus,
\[
|I_{\Omega_1,\omega_1}(f)(x)| \leq C \left| \int f \right|_{L_\nu^{\alpha}(\nu(x),\nu')} \sum_{k=1}^{\infty} \omega^\alpha \left( 2^{k+1}B \right)^{\nu/\nu'}
\]
Therefore, it can be obtained that
\[
I_2 \leq C \left| \int f \right|_{L_\nu^{\alpha}(\nu(x),\nu')} \sum_{k=1}^{\infty} \omega^\alpha \left( 2^{k+1}B \right)^{\nu/\nu'}
\]
Noting that
\[
\omega^\alpha = \nu^\alpha \in A_{\nu/\nu'},
\]
there exists a constant \(s > 1\), such that \(\nu^\alpha \in RH_s\). From Lemma 2.3, it can be obtained that
\[
\frac{\omega^\alpha(B)}{\omega^\alpha(2^{k+1}B)} \leq C \left( \frac{B}{2^{k+1}B} \right)^{(s-1)/p}
\]
Finally, since \(s > 1\) and \(0 < \kappa < p/q\), we see that
\[
I_2 \leq C \left| \int f \right|_{L_\nu^{\alpha}(\nu(x),\nu')} \sum_{k=1}^{\infty} \left( \frac{1}{2^{k+1}} \right)^{(1-\nu)(q-k)/p} \leq C \left| \int f \right|_{L_\nu^{\alpha}(\nu(x),\nu')}
\]
Hence, (1) is proved. The proof of Theorem 1.1 is now completed.

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